

# Methoden moderner Röntgenphysik: Streuung und Abbildung

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Lecture 9	Vorlesung zum Haupt- oder Masterstudiengang Physik, SoSe 2019 G. Grübel, L. Müller, O. Seeck, L. Frenzel, F. Lehmkuhler, M. Martins, W. Wurth		
Location	Lecture hall AP, Physics, Jungiusstraße		
Date	Tuesdays	12:30 - 14:00	(starting 2.4.)
	Thursdays	8:30 - 10:00	(until 11.7.)



# Methoden moderner Röntgenphysik II: Streuung und Abbildung

## Small Angle Scattering, and Soft Matter

Introduction, Form Factor, Structure Factor, Applications, ...

## Anomalous Diffraction

Introduction into Anomalous Scattering, ...

## **Introduction into Coherence**

**Concept, First Order Coherence, ...**

## Coherent Scattering, Applications of Coherent Scattering

Spatial Coherence, Second Order Coherence,  
Imaging and Correlation Spectroscopy, ...



# The Concept of Coherence: Classical Light

First Order Coherence

Coherence and Emission Spectrum

Spatial Coherence

Second Order Coherence

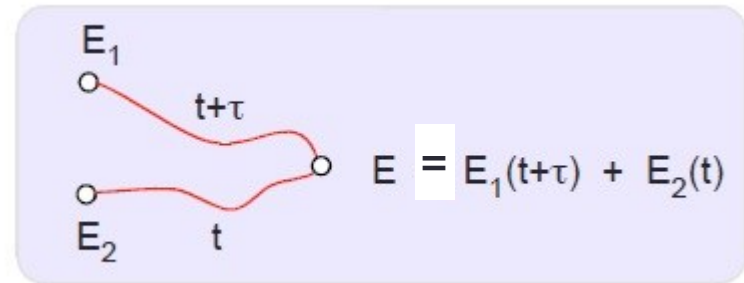
Chaotic Light

Basic concepts: **The quantum theory of light**  
Rodney Loudon, Oxford University Press (1990)  
**Quantum optics**  
Marlan O. Scully, M. Suhail Zubairy,  
Cambridge University Press (1997)



# The Concept of Coherence

Consider harmonic fields  $E_1, E_2$  at positions  $r_1, r_2$  at time  $t=0$ :



$$\langle I_n \rangle \equiv \langle E_n(t)E_n^*(t) \rangle, \quad n \in \{1, 2\}$$

$$\langle I \rangle = \langle EE^* \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2 \operatorname{Re} [\langle E_1(t + \tau)E_2^*(t) \rangle] \quad \text{with } \langle f \rangle \equiv \lim_{T \rightarrow \infty} \langle f \rangle_T$$

$$\langle f \rangle_T \equiv \left( \frac{1}{T} \right) \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt$$

Here  $\lim T \rightarrow \infty$  means that  $T$  is finite but sufficiently large such that  $\langle f \rangle_T$  does not depend on  $T$

Normalized pair correlation function: 
$$\gamma_{12}(\tau) \equiv \frac{\langle E_1(t + \tau)E_2^*(t) \rangle}{(\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}}}$$

$$\Rightarrow \langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2 (\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}} \operatorname{Re}[\gamma_{12}(\tau)]$$

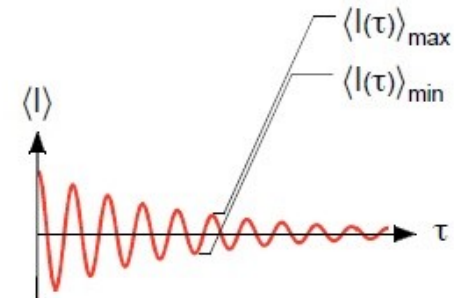
$$\gamma_{12}(\tau) = |\gamma_{12}(\tau)| \cos(\phi_{12}(\tau)) \Rightarrow I = \langle I_1 \rangle + \langle I_2 \rangle + 2(\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}} |\gamma_{12}(\tau)| \cos(\phi_{12}(\tau))$$

Assume:  $\phi_{12}(\tau)$  changes much faster than  $|\gamma_{12}(\tau)|$  (quasi-coherent light)

$$\Rightarrow \langle I \rangle_{max/min} = \langle I_1 \rangle + \langle I_2 \rangle \pm 2(\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}} |\gamma_{12}(\tau)|$$

Interference visibility  $\kappa$  :

$$\kappa \equiv \left| \frac{\langle I \rangle_{max} - \langle I \rangle_{min}}{\langle I \rangle_{max} + \langle I \rangle_{min}} \right| = 2 \frac{(\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}}}{\langle I_1 \rangle + \langle I_2 \rangle} |\gamma_{12}(\tau)|$$



$$\langle I_1 \rangle = \langle I_2 \rangle \Rightarrow \kappa = |\gamma_{12}(\tau)|$$

Definition:	$ \gamma_{12}(\tau)  = 1$	for all $\tau$	$\Rightarrow$	complete coherence
	$0 <  \gamma_{12}(\tau)  < 1$	for some $\tau$	$\Rightarrow$	partial coherence
	$ \gamma_{12}(\tau)  = 0$	for all $\tau$	$\Rightarrow$	no coherence

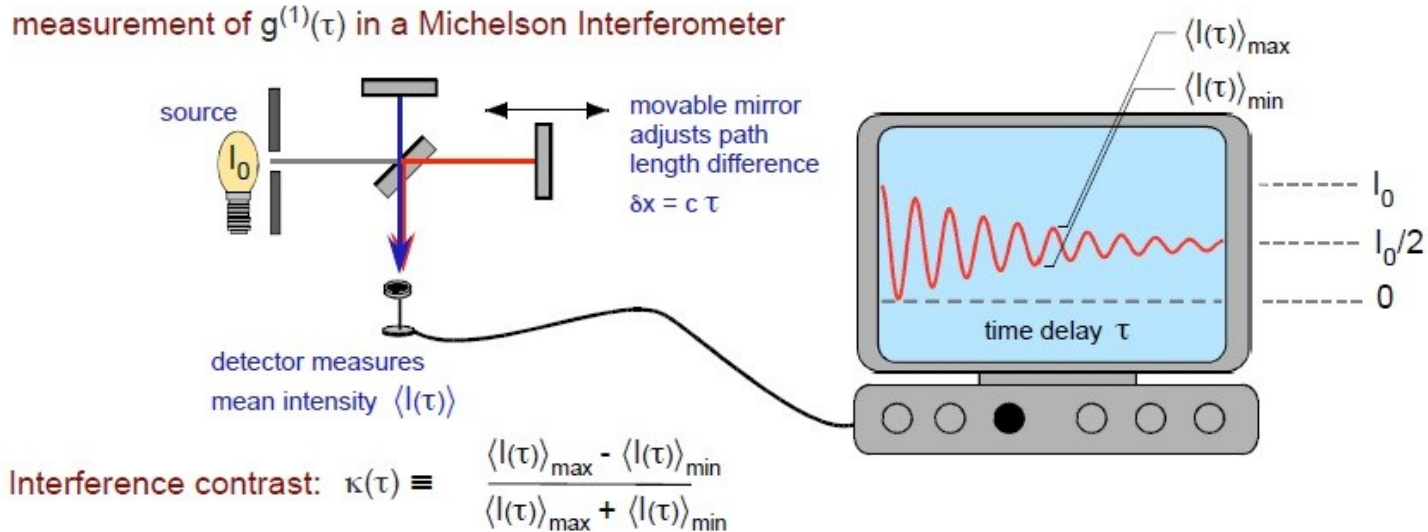
Normalized autocorrelation function:

$$g^{(1)}(\tau) \equiv \frac{\langle E(t+\tau)E^*(t) \rangle}{\langle I \rangle}$$

with  $g^{(1)}(0) = 1$  and  $g^{(1)}(-\tau) = g^{(1)*}(\tau)$

Measurement of  $g^{(1)}(\tau)$  in a Michelson Interferometer

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Maximal coherence:

Interference contrast maximal for all  $\tau$



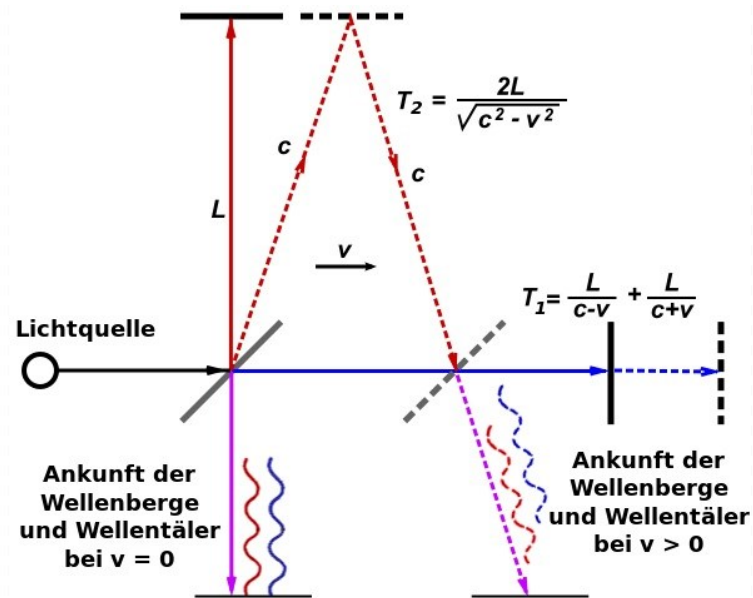
Partial coherence:

Interference contrast decreases for large  $\tau$



# The Michelson-Morley Experiment

by Albert Abraham Michelson: 1881 in Potsdam



Normalized autocorrelation function:

$$g^{(1)}(\tau) \equiv \frac{\langle E(t+\tau)E^*(t) \rangle}{\langle I \rangle}$$

with  $g^{(1)}(0) = 1$  and  $g^{(1)}(-\tau) = g^{(1)*}(\tau)$

Example: successive wave trains of duration  $\tau_0$  and length  $c\tau_0$

$$E(t) = E_0 e^{i\omega t + i\phi(t)} \quad \text{with } \phi(t) \text{ (see figure)}$$

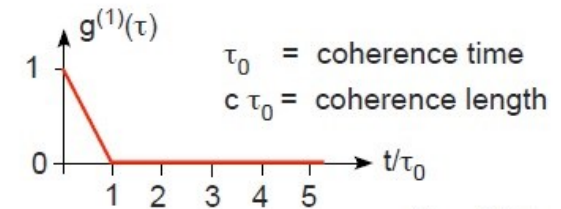
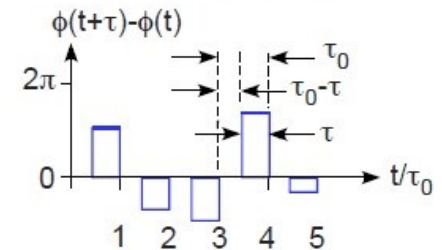
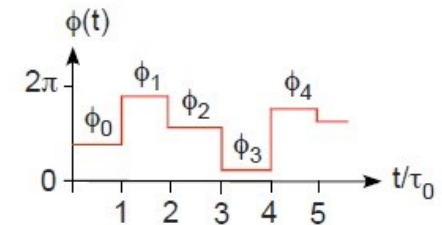
$$\Rightarrow g^{(1)}(\tau) = e^{i\omega\tau} \langle e^{i(\phi(t+\tau) - \phi(t))} \rangle$$

$0 \leq \tau \leq \tau_0$ :

$$\langle e^{i(\phi(t+\tau) - \phi(t))} \rangle = \frac{1}{N\tau_0} \sum_{n=0}^{N+1} \int_{n\tau_0}^{(n+1)\tau_0} dt e^{i(\phi(t+\tau) - \phi(t))}$$

$$= \frac{1}{N\tau_0} \sum_{n=0}^{N+1} \{(\tau_0 - \tau) + \tau e^{i(\phi_{n+1} - \phi_n)}\}$$

$$\Rightarrow g^{(1)}(\tau) = e^{i\omega\tau} \begin{cases} \frac{\tau_0 - \tau}{\tau_0} & \times 0 & \text{if } \tau_0 < \tau \\ 1 & \times 1 & \text{if } 0 \leq \tau \leq \tau_0 \end{cases}$$



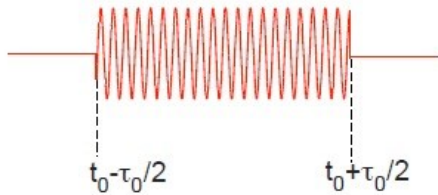
**Note:**  $\tau_0$ : coherence time;  $\xi_l = \frac{\lambda}{2 \Delta\lambda} = c\tau_0$ : longitudinal coherence length





# Coherence and Emission Spectrum

Consider single wave train of duration  $\tau_0$ , phase  $\phi_0$ , frequency  $\omega_0$ :

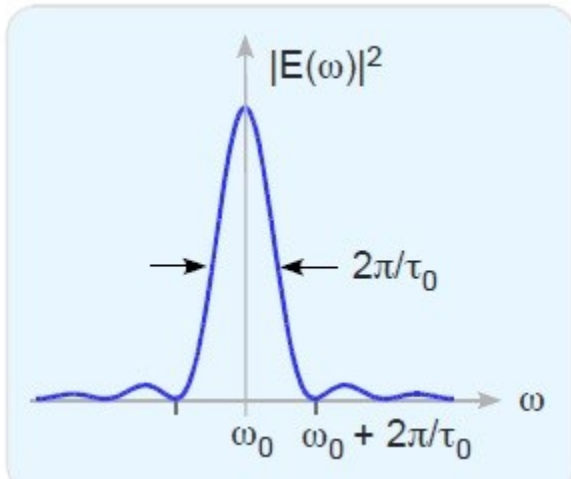


$$E(t) = e^{[-i\omega_0 t - i\phi_0]} \times 1 \quad \left(\text{if } \frac{\tau_0}{2} \leq t \leq \frac{\tau_0}{2}\right)$$

$$\times 0 \quad \text{otherwise}$$

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt E(t) e^{i\omega t} = \sqrt{\frac{2}{\pi}} \frac{\sin\left(\frac{(\omega - \omega_0)\tau_0}{2}\right)}{(\omega - \omega_0)} \cdot e^{-i\phi_0}$$

N wave trains with the same frequency  $\omega_0$  but arbitrary phases  $\phi_n$ :



$$|E(\omega)|^2 = \sum_{n=1}^N |E_n(\omega)|^2 = \frac{2}{\pi} \sum_{n=1}^N \frac{\sin^2\left(\frac{(\omega - \omega_0)\tau_n}{2}\right)}{(\omega - \omega_0)^2}$$

Emission bandwidth  $\Delta\nu \cong \frac{1}{\tau}$  with  $\tau \equiv \frac{1}{N} \sum_{n=1}^N \tau_n$

# Example: Collision Broadened Light Source

Molecules of a gas radiate light  $E(t) = E_0 e^{-i(\omega_0 t - \phi(t))}$  at frequency  $\omega_0$ . Collisions yield random phase jumps, i.e., phase  $\phi(t) \in [0, 2\pi]$  fluctuates.

Probability for a free flight of duration  $t \in [\tau, \tau + d\tau]$ :  $P(t) = \frac{1}{\tau_0} \exp(-\tau/\tau_0)$   
 kinetic gas theory ( $\tau_0$  means duration of free flight)

Coherence function:  $g^{(1)}(\tau) = e^{i\omega_0 \tau} \langle e^{i(\phi(t+\tau) - \phi(t))} \rangle$

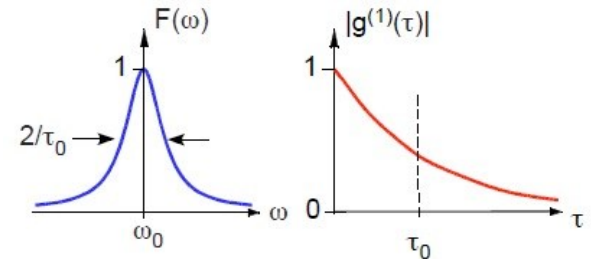
$$\begin{aligned}
 e^{i(\phi(t+\tau) - \phi(t))} &= 1 && \text{for free flights with duration } > \tau \\
 &= e^{i\chi} && \text{with } \chi \in [0, 2\pi] \text{ random for free flights with} \\
 &&& \text{duration } < \tau
 \end{aligned}$$

i.e., only flights of duration  $t > \tau$  yield contribution to  $\langle e^{i(\phi(t+\tau) - \phi(t))} \rangle$ :

$$\Rightarrow g^{(1)}(\tau) = e^{i\omega_0 \tau} \int_0^\infty P(s) ds = e^{i\omega_0 \tau} \exp(-\tau/\tau_0)$$

$$\Rightarrow |g^{(1)}(\tau)| = \exp(-\tau/\tau_0)$$

$$\Rightarrow F(\omega) = \frac{1}{1 + (\omega - \omega_0)^2 \tau_0^2} \quad \text{Wiener-Khinchin Theorem}$$



# Wiener Khinchin Theorem

$$E(\omega) \equiv \mathcal{F}[E(t)] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt E(t) e^{i\omega t}$$

$$F(\omega) \equiv \frac{|E(\omega)|^2}{\int_{-\infty}^{\infty} dt |E(\omega)|^2}$$

normalized spectral density

$$\Rightarrow F(\omega) = \frac{1}{\sqrt{2\pi}} \mathcal{F}[g^{(1)}],$$

$\mathcal{F} \equiv$  Fourier-Transform

## Wiener Khinchin Theorem

The spectral power density  $F(\omega)$  is the Fourier transform of the normalized autocorrelation function  $g^{(1)}(\tau)$

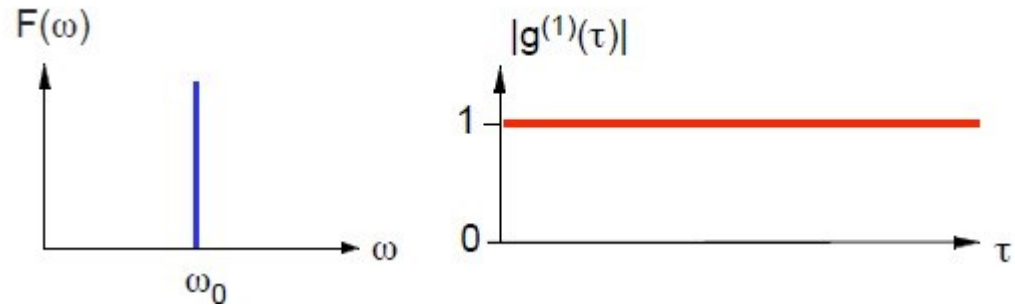


# Example: Monochromatic Light

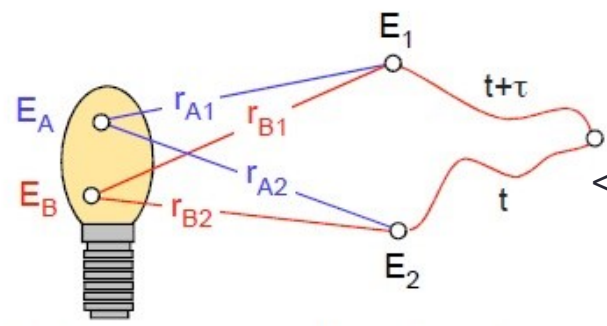
$$E(t) = e^{-i(\omega_0 t - \phi)}$$

$$g^{(1)}(\tau) = e^{i\omega_0 \tau}$$

$$|g^{(1)}(\tau)| = 1$$



# Spatial Coherence



Light Source: mutually incoherent point sources  $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

$$E_1 = E_{A1} + E_{B1} \quad E_{An} = E_A e^{\frac{ir_{An}\omega}{c}}$$

$$E_2 = E_{A2} + E_{B2} \quad E_{Bn} = E_B e^{\frac{ir_{Bn}\omega}{c}}$$

$$\langle E_1(t + \tau)E_2^*(t) \rangle = \langle E_{A1}(t + \tau)E_{A2}^*(t) \rangle + \langle E_{B1}(t + \tau)E_{B2}^*(t) \rangle + \langle E_{A1}(t + \tau)E_{B2}^*(t) \rangle + \langle E_{B1}(t + \tau)E_{A2}^*(t) \rangle$$

$$\langle I_n \rangle = \langle E_n(t)E_n^*(t) \rangle = \langle E_{An}(t)E_{An}^*(t) \rangle + \langle E_{Bn}(t)E_{Bn}^*(t) \rangle + \langle E_{An}(t)E_{Bn}^*(t) \rangle + \langle E_{Bn}(t)E_{An}^*(t) \rangle$$

$$\Rightarrow \langle I_1 \rangle = \langle I_2 \rangle$$

$$\langle E_{A1}(t + \tau)E_{A2}^*(t) \rangle = \langle E_A(t + \tau)E_A^*(t) \rangle e^{\left[ \frac{i(r_{A1} - r_{A2})\omega}{c} \right]} = \langle E_A(t + \tau_A)E_A^*(t) \rangle \text{ with } \tau_A \equiv \tau + \frac{(r_{A1} - r_{A2})}{c}$$

$$\langle E_{B1}(t + \tau)E_{B2}^*(t) \rangle = \langle E_B(t + \tau)E_B^*(t) \rangle e^{\left[ \frac{i(r_{B1} - r_{B2})\omega}{c} \right]} = \langle E_B(t + \tau_B)E_B^*(t) \rangle \text{ with } \tau_B \equiv \tau + \frac{(r_{B1} - r_{B2})}{c}$$

$$\Rightarrow \langle E_1(t + \tau)E_2^*(t) \rangle = \langle E_A(t + \tau_A)E_A^*(t) \rangle + \langle E_B(t + \tau_B)E_B^*(t) \rangle$$

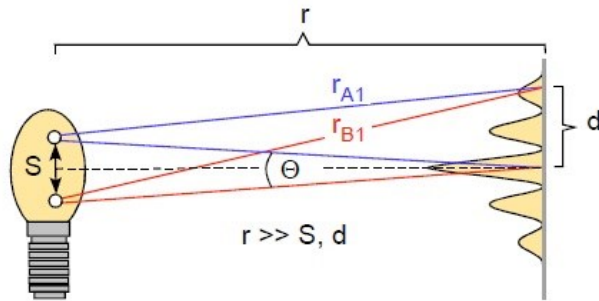
$$\gamma_{12}(\tau) \equiv \frac{\langle E_1(t + \tau)E_2^*(t) \rangle}{(\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}}} = \frac{1}{2} [g^{(1)}(\tau_A) + g^{(1)}(\tau_B)] = \frac{1}{2} \left[ e^{i\omega\tau_A - \frac{\tau_A}{\tau_0}} + e^{i\omega\tau_B - \frac{\tau_B}{\tau_0}} \right]$$

$$4|\gamma_{12}(\tau)|^2 \equiv |g^{(1)}(\tau_A)|^2 + |g^{(1)}(\tau_B)|^2 + 2|g^{(1)}(\tau_A)||g^{(1)}(\tau_B)|\cos(\omega(\tau_A - \tau_B)) \text{ interference term}$$



$$4|\gamma_{12}(\tau)|^2 \equiv |g^{(1)}(\tau_A)|^2 + |g^{(1)}(\tau_B)|^2 + 2|g^{(1)}(\tau_A)| |g^{(1)}(\tau_B)| \cos(\omega(\tau_A - \tau_B))$$

Interference term depends on  $\tau_A - \tau_B = \frac{r_{A1} - r_{A2}}{c} - \frac{r_{B1} - r_{B2}}{c}$



Light Source: mutually incoherent point sources  $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

symmetric:  $r_{A2} - r_{B2} = 0 \Rightarrow \tau_A - \tau_B = \frac{r_{A1} - r_{B1}}{c}$

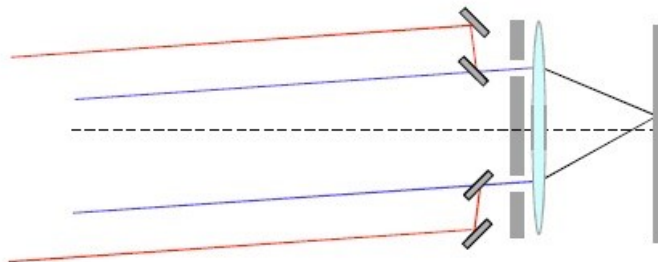
$$r_{A1} \cong r + \frac{(d-S)^2}{2r}, \quad r_{B1} \cong r + \frac{(d+S)^2}{2r}$$

$$\Rightarrow \tau_A - \tau_B \cong -\frac{Sd}{2rc}$$

First minimum of  $|\gamma_{12}(\tau)|^2$  :

$$\omega(\tau_A - \tau_B) = \pi; \quad S \cong r\theta \quad \Rightarrow \quad d \cong \frac{\lambda}{\theta}$$

**transverse coherence length**



Michelson stellar interferometer: adjustable slits, extension of slit separation by mirrors

Measurement of angular diameter of stars, angular separation of double stars, etc.