Methoden moderner Röntgenphysik II: Streuung und Abbildung

Lecture 9 — Vorlesung zum Haupt- oder Masterstudiengang Physik, SoSe 2016

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Location — Lecture hall AP, Physics, Jungiusstraße

Date	Tuesday	12:30 - 14:00
	Thursday	8:30 - 10:00

Methoden moderner Röntgenphysik II: Streuung und Abbildung

Small Angle Scattering, and Soft Matter
Introduction, Form Factor, Structure Factor, Applications, ...

Anomalous Diffraction
Introduction into Anomalous Scattering, ...

Introduction into Coherence
Concept, First Order Coherence, ...

Coherent Scattering
Spatial Coherence, Second Order Coherence, ...

Applications of Coherent Scattering
Imaging and Correlation Spectroscopy, ...

The Concept of Coherence: Classical Light

First Order Coherence

Coherence and Emission Spectrum

Spatial Coherence

Second Order Coherence

Chaotic Light

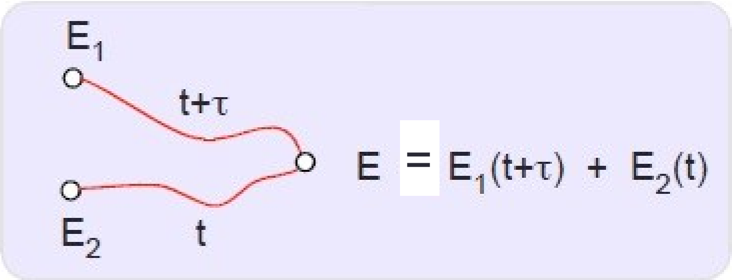
Basic concepts:

The quantum theory of light
Rodney Loudon, Oxford University Press (1990)

Quantum optics
Marlan O. Scully, M. Suhal Zubairy,
Cambridge University Press (1997)

The Concept of Coherence

Consider harmonic fields E_1, E_2 at positions r_1, r_2 at time:



$$< I_n > \equiv < E_n(t)E_n^*(t) >, \qquad n \in \{1,2\}$$

$$< I > = < EE^* > = < I_1 > + < I_2 > + 2\,\mathrm{Re}\,[< E_1(t+\tau)E_2^*(t) >] \qquad \text{with } < f > \equiv \lim_{T\to\infty} < f >_T$$

$$< f >_T \equiv \left(\frac{1}{T}\right) \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$$

Here lim T→∞ means that T is finite but sufficiently large such that <f>$_T$ does not depend on T

Normalized pair correlation function: $\gamma_{12}(\tau) \equiv \dfrac{<E_1(t+\tau)E_2^*(t)>}{(<I_1><I_2>)^{\frac{1}{2}}}$

$$\Rightarrow < I > = < I_1 > + < I_2 > + 2\,(< I_1 >< I_2 >)^{\frac{1}{2}}\ \mathrm{Re}[\gamma 12(\tau)]$$

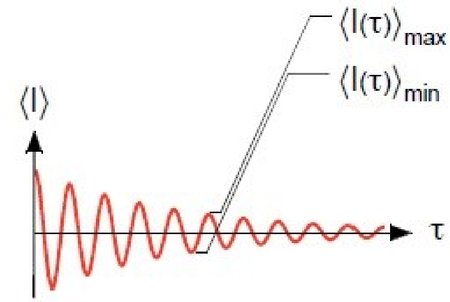
$$\gamma_{12}(\tau) = |\gamma_{12}(\tau)|\ \cos(\phi_{12}(\tau)) \Rightarrow I = \langle I_1 \rangle + \langle I_2 \rangle + 2(\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}}\ |\gamma_{12}(\tau)| \cos(\phi_{12}(\tau))$$

Assume: $\phi_{12}(\tau)$ changes much faster than $|\gamma_{12}(\tau)|$ (quasi-coherent light)

$$\Rightarrow \langle I \rangle\, max/min = \langle I_1 \rangle + \langle I_2 \rangle \pm 2(\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}}\ |\gamma_{12}(\tau)|$$

Interference visibility κ :

$$\kappa \equiv \left| \frac{\langle I \rangle_{max} - \langle I \rangle_{min}}{\langle I \rangle_{max} + \langle I \rangle_{min}} \right| = 2 \frac{(\langle I_1 \rangle \langle I_2 \rangle)^{\frac{1}{2}}}{\langle I_1 \rangle + \langle I_2 \rangle}\ |\gamma_{12}(\tau)|$$



$$\langle I_1 \rangle = \langle I_2 \rangle \Rightarrow \kappa = |\gamma_{12}(\tau)|$$

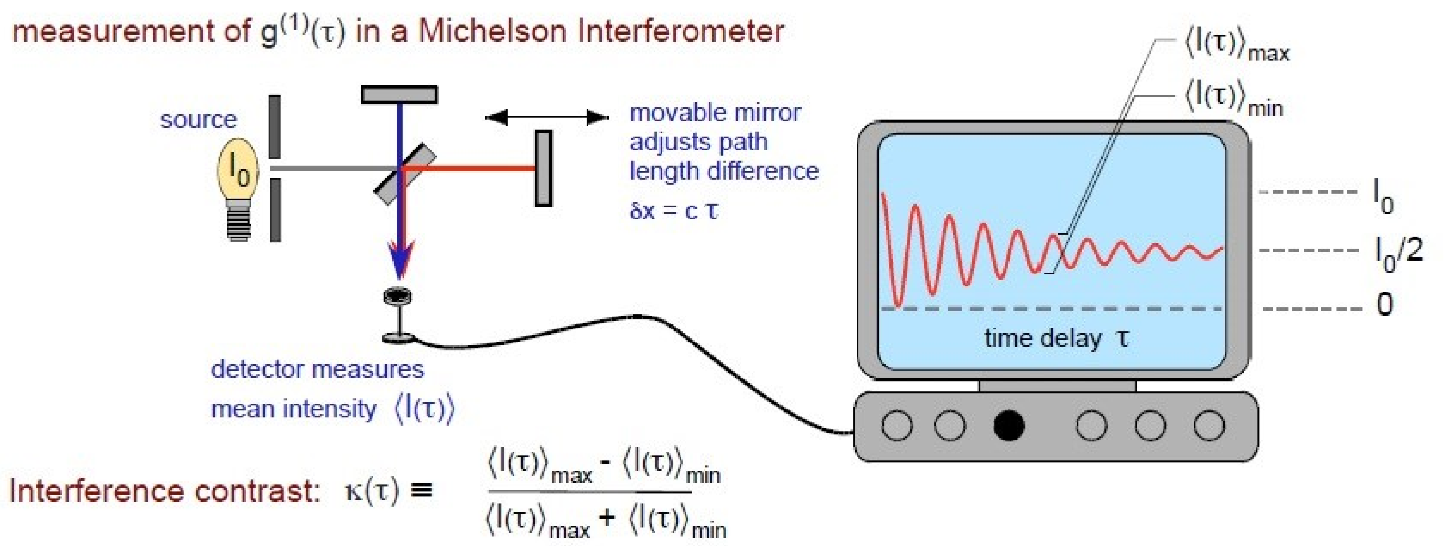
Definition:	$|\gamma_{12}(\tau)| = 1$	for all τ	$\Rightarrow$	complete coherence
	$0 < |\gamma_{12}(\tau)| < 1$	for some τ	$\Rightarrow$	partial coherence
	$|\gamma_{12}(\tau)| = 0$	for all τ	$\Rightarrow$	no coherence

Normalized autocorrelation function:

$$g^{(1)}(\tau) \equiv \frac{<E(t+\tau)E^*(t)>}{<I>}$$

$$\text{with } g^{(1)}(0) = 1 \text{ and } g^{(1)}(-\tau) = g^{(1)*}(\tau)$$

Measurement of $g^{(1)}(\tau)$ in a Michelson Interferometer

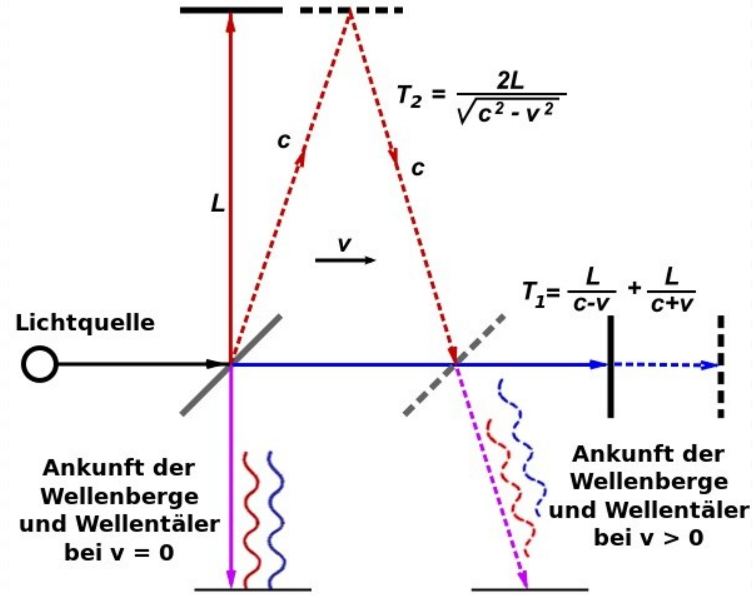


Maximal coherence: Interference contrast maximal for all τ 

Partial coherence: Interference contrast decreases for large τ 

The Michelson-Morley Experiment

by Albert Abraham Michelson: 1881 in Potsdam



Normalized autocorrelation function: $g^{(1)}(\tau) \equiv \frac{<E(t+\tau)E^*(t)>}{<I>}$

with $g^{(1)}(0) = 1$ and $g^{(1)}(-\tau) = g^{(1)*}(\tau)$

Example: successive wave trains of duration τ_0 and length $c\tau_0$

$E(t) = E_0 e^{i\omega t + i\phi(t)}$ with $\phi(t)$ (see figure)

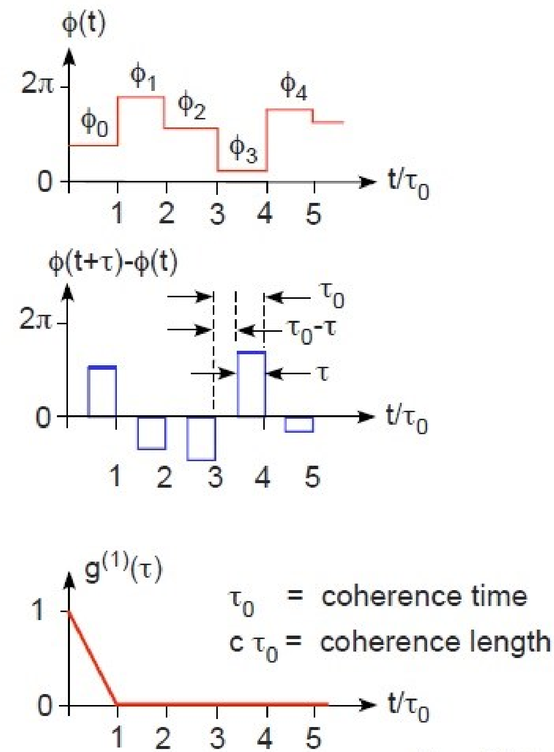
$\Rightarrow g^{(1)}(\tau) = e^{i\omega\tau} < e^{i(\phi(t+\tau)-\phi(t))} >$

$0 \leq \tau \leq \tau_0$:

$$< e^{i(\phi(t+\tau)-\phi(t))} > = \frac{1}{N\tau_0} \sum_{n=0}^{N+1} \int_{n\tau_0}^{(n+1)\tau_0} dt\, e^{i(\phi(t+\tau)-\phi(t))}$$

$$= \frac{1}{N\tau_0} \sum_{n=0}^{N+1} \{(\tau_0 - \tau) + \tau\, e^{i(\phi_{n+1} - \phi_n)}\}$$

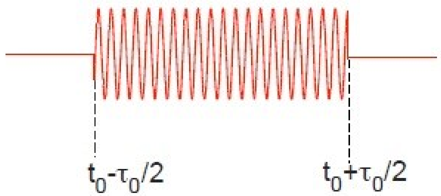
$$\Rightarrow g^{(1)}(\tau) = e^{i\omega t} \frac{(\tau_0 - \tau)}{\tau_0} \quad \begin{matrix} x\ 0 & \text{if } \tau_0 < \tau \\ x\ 1 & \text{if } 0 \leq \tau \leq \tau_0 \end{matrix}$$



Note: τ_0: **coherence time;** $\xi_l = \frac{\lambda}{2}\frac{\lambda}{\Delta\lambda} = c\tau_0$: **longitudinal coherence length**

Coherence and Emission Spectrum

Consider single wave train of duration τ_0, phase ϕ_0, frequency ω_0:

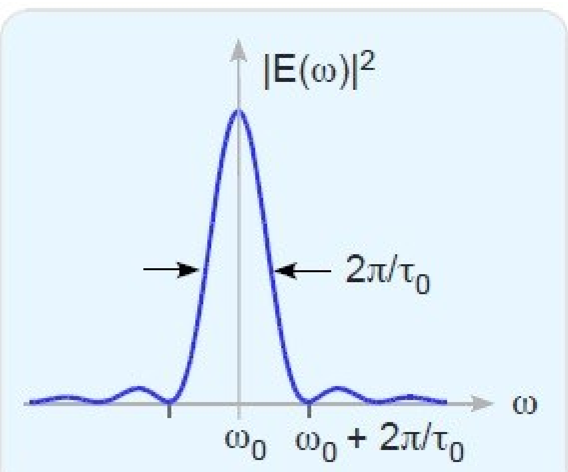


$$E(t) = e^{[-i\omega_0 t - i\phi_0]} \times 1 \qquad (\text{if } \frac{\tau_0}{2} \leq \tau \leq \frac{\tau_0}{2})$$

$$\times 0 \qquad \text{otherwise}$$

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\; E(t)\; e^{i\omega t} = \sqrt{\frac{2}{\pi}} \frac{\sin\left(\frac{(\omega-\omega_0)\tau_0}{2}\right)}{(\omega-\omega_0)} \bullet e^{-i\phi_0}$$

N wave trains with the same frequency ω_0 but arbitrary phases ϕ_n:



$$|E(\omega)|^2 = \sum_{n=1}^{N} |E_n(\omega)|^2 = \frac{2}{\pi} \sum_{n=1}^{N} \frac{\sin^2\left(\frac{(\omega-\omega_0)\tau_n}{2}\right)}{(\omega-\omega_0)^2}$$

Emission bandwidth $\Delta\nu \cong \frac{1}{\tau}$ with $\tau \equiv \frac{1}{N}\sum_{n=1}^{N} \tau_n$

Example: Collision Broadened Light Source

Molecules of a gas radiate light $E(t) = E_0 e^{-i(\omega_0 t - \phi(t))}$ at frequency ω_0. Collisions yield random phase jumps, i.e., phase $\phi(t) \in [0, 2\pi]$ fluctuates.

Probability for a free flight of duration $t \in [\tau, \tau + d\tau]$: $P(t) = \frac{1}{\tau_0} \exp(\tau/\tau_0)$

kinetic gas theory (τ_0 means duration of free flight)

Coherence function: $g^{(1)}(\tau) = e^{i\omega_0\tau} < e^{i(\phi(t+\tau) - \phi(t))} >$

$e^{i(\phi(t+\tau) - \phi(t))}$ = 1 for free flights with duration > τ

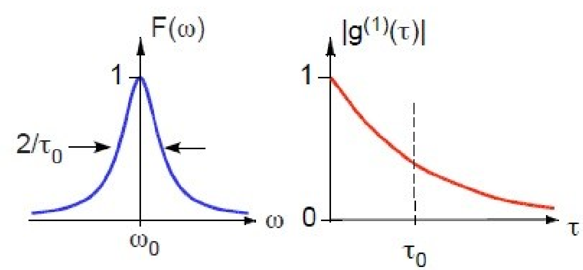
= $e^{i\chi}$ with $\chi \in [0, 2\pi]$ random for free flights with duration < τ

i.e., only flights of duration t > τ yield contribution to $< e^{i(\phi(t+\tau) - \phi(t))} >$:

$\Rightarrow \quad g^{(1)}(\tau) = e^{i\omega\tau} \int_\tau^\infty P(s)ds = e^{i\omega\tau} \exp(-\tau/\tau_0)$

$\Rightarrow \quad |g^{(1)}(\tau)| = \exp(\tau/\tau 0)$

$\Rightarrow \quad F(\omega) = \frac{1}{1+(\omega-\omega_0)^2\tau_0^2}$ **Wiener-Khinchin Theorem**



Wiener Khinchin Theorem

$$E(\omega) \equiv \mathcal{F}[E(t)] \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\ E(t)\ e^{i\omega t}$$

$$F(\omega) \equiv \frac{|E(\omega)|^2}{\int_{-\infty}^{\infty} dt |E(\omega)|^2}$$

normalized spectral density

$$\Rightarrow F(\omega) = \frac{1}{\sqrt{2\pi}} \mathcal{F}\left[g^{(1)}\right],$$

$\mathcal{F} \equiv$ Fourier-Transform

Wiener Khinchin Theorem

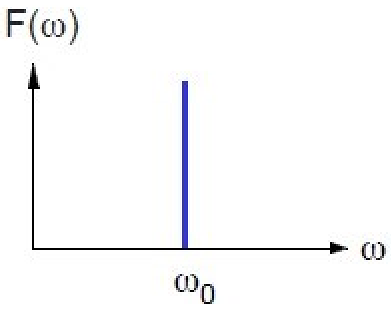
The spectral power density $F(\omega)$ is the Fourier transform of the normalized autocorrelation function $g^{(1)}(\tau)$

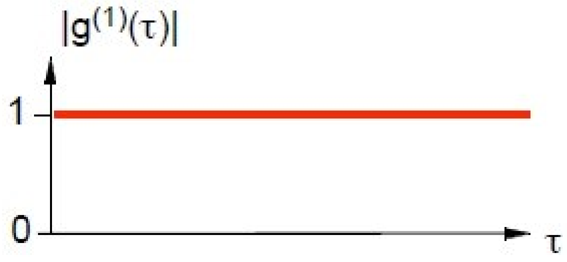
Example: Monochromatic Light

$$E(t) = e^{-i(\omega_0 t - \phi)}$$

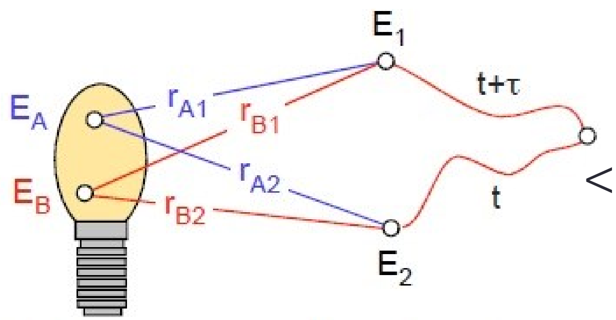
$$g^{(1)}(\tau) = e^{i\omega_0 \tau}$$

$$\left|g^{(1)}(\tau)\right| = 1$$





Spatial Coherence



Light Source: mutually incoherent point sources $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

$$E_1 = E_{A1} + E_{B1} \qquad E_{An} = E_A e^{\frac{i r_{An}\omega}{c}}$$

$$E_2 = E_{A2} + E_{B2} \qquad E_{Bn} = E_B e^{\frac{i r_{Bn}\omega}{c}}$$

$$< E_1(t+\tau)E_2^*(t) > = < E_{A1}(t+\tau)E_{A2}^*(t) > + < E_{B1}(t+\tau)E_{B2}^*(t) > + < E_{A1}(t+\tau)E_{B2}^*(t) > + < E_{B1}(t+\tau)E_{A2}^*(t) >$$

$$< I_n > = < E_n(t)E_n^*(t) > = < E_{An}(t)E_{An}^*(t) > + < E_{Bn}(t)E_{Bn}^*(t) > + < E_{An}(t)E_{Bn}^*(t) > + < E_{Bn}(t)E_{An}^*(t) >$$

$$\Rightarrow < I_1 > = < I_2 >$$

$$< E_{A1}(t+\tau)E_{A2}^*(t) > = < E_A(t+\tau)E_A^*(t) > e^{\left[\frac{i(r_{A1}-r_{A2})\omega}{c}\right]} = < E_A(t+\tau_A)E_A^*(t) > \text{ with } \tau_A \equiv \tau + \frac{(r_{A1}-r_{A2})}{c}$$

$$< E_{B1}(t+\tau)E_{B2}^*(t) > = < E_B(t+\tau)E_B^*(t) > e^{\left[\frac{i(r_{B1}-r_{B2})\omega}{c}\right]} = < E_B(t+\tau_B)E_B^*(t) > \text{ with } \tau_B \equiv \tau + \frac{(r_{B1}-r_{B2})}{c}$$

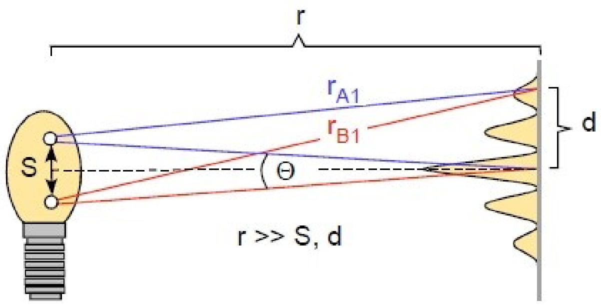
$$\Rightarrow < E_1(t+\tau)E_2^*(t) > = < E_A(t+\tau_A)E_A^*(t) > + < E_B(t+\tau_B)E_B^*(t) >$$

$$\gamma_{12}(\tau) \equiv \frac{< E_1(t+\tau)E_2^*(t) >}{(< I_1 >< I_2 >)^{\frac{1}{2}}} = \frac{1}{2}\left[g^{(1)}(\tau_A) + g^{(1)}(\tau_B)\right] = \frac{1}{2}\left[e^{i\omega\tau_A - \frac{\tau_A}{\tau_0}} + e^{i\omega\tau_B - \frac{\tau_B}{\tau_0}}\right]$$

$$4|\gamma_{12}(\tau)|^2 \equiv \left|g^{(1)}(\tau_A)\right|^2 + \left|g^{(1)}(\tau_B)\right|^2 + 2\left|g^{(1)}(\tau_A)\right|\left|g^{(1)}(\tau_B)\right|\cos(\omega(\tau_A - \tau_B))$$ **interference term**

$$4|\gamma_{12(\tau)}|^2 \equiv \left|g^{(1)}(\tau_A)\right|^2 + \left|g^{(1)}(\tau_B)\right|^2 + 2\left|g^{(1)}(\tau_A)\right| \left|g^{(1)}(\tau_B)\right| \cos(\omega(\tau_A - \tau_B))$$

Interference term depends on $\tau_A - \tau_B = \frac{r_{A1}-r_{A2}}{c} - \frac{r_{B1}-r_{B2}}{c}$



Light Source: mutually incoherent point sources $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

symmetric: $r_{A2} = r_{B2} = 0 \Rightarrow \tau_A - \tau_B = \frac{r_{A1} - r_{B1}}{c}$

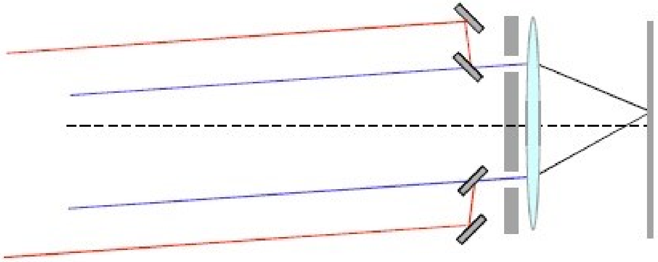
$$r_{A1} \cong r + \frac{\left(\frac{d-S}{2}\right)^2}{2r}, \; r_{B1} \cong r + \frac{\left(\frac{d+S}{2}\right)^2}{2r}$$

$$\Rightarrow \tau_A - \tau_B \cong -\frac{Sd}{2rc}$$

First minimum of $|\gamma_{12}(\tau)|^2$:

$\omega(\tau_A - \tau_B) = \pi$; $S \cong r\theta$ $\Rightarrow$ $d \cong \frac{\lambda}{\theta}$

transverse coherence length



Michelson stellar interferometer: adjustable slits, extension of slit separation by mirrors

Measurement of angular diameter of stars, angular separation of double stars, etc.