Methoden moderner Röntgenphysik II: Streuung und Abbildung

Lecture 9

Vorlesung zum Haupt/Masterstudiengang Physik

SS 2014

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Location: Hörs AP, Physik, Jungiusstrasse

Tuesdays 12.45 – 14.15

Thursdays 8:30 – 10.00

Methoden moderner Röntgenphysik II: Streuung und Abbildung

Small Angle Scattering, and Soft Matter
Introduction, form factor, structure factor, applications, ..

Anomalous Diffraction
Introduction into anomalous scattering,..

Introduction into Coherence
Concept, First order coherence, ..

Coherent Scattering
Spatial coherence, second order coherence,..

Applications of coherent Scattering
Imaging and Correlation spectroscopy,..

The concept of coherence: classical light

First order coherence

Coherence and emission spectrum

Spatial coherence

Second order coherence

Chaotic light

Basic concepts:

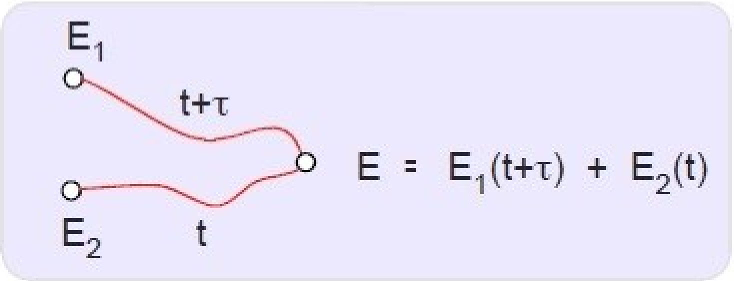
The quantum theory of light
Rodney Loudon, Oxford University Press (1990)

Quantum optics
Marlan O. Scully, M. Suhal Zubairy,
Cambridge University Press (1997)

Courtesy: Andreas Hemmerich

The concept of coherence

Consider harmonic fields E_1, E_2 at positions r_1, r_2 at time:



$\langle I_n \rangle \equiv \langle E_n(t)\, E_n^*(t) \rangle$, $\quad n \in \{1, 2\}$

$\langle I \rangle = \langle EE^* \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\,\mathrm{Re}\,[\langle E_1(t+\tau)\, E_2^*(t) \rangle]$ with $\langle f \rangle \equiv \lim_{T\to\infty} \langle f \rangle_T$

$\langle f \rangle_T \equiv (1/T) \int_{-T/2}^{T/2} f(t)\, dt$

here the limes $T \to \infty$ means that T is finite but sufficiently large such that $\langle f \rangle_T$ does not depend on T

Normalized pair correlation function: $\gamma_{12}(\tau) \equiv \langle E_1(t+\tau)\, E^*_2(t) \rangle / (\langle I_1 \rangle \langle I_2 \rangle)^{1/2}$

$\Rightarrow \langle I \rangle = \langle I_1 \rangle + \langle I_2 \rangle + 2\, (\langle I_1 \rangle \langle I_2 \rangle)^{1/2}\, \mathrm{Re}[\gamma_{12}(\tau)]$

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$\gamma_{12}(\tau) = |\gamma_{12}(\tau)| \exp(i\phi_{12}(\tau)) \Rightarrow I = <I_1> + <I_2> + 2(<I_1><I_2>)^{1/2} |\gamma_{12}(\tau)| \cos(\phi_{12}(\tau))$

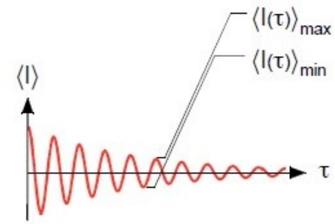
Assume: $\phi_{12}(\tau)$ changes much faster than $|\gamma_{12}(\tau)|$ (quasi-coherent light)

$\Rightarrow <I>_{max/min} = <I_1> + <I_2> +/- 2(<I_1><I_2>)^{1/2} |\gamma_{12}(\tau)|$

Interference visibility:

$\kappa \equiv |(<I>_{max} - <I>_{min} / <I>_{max} + <I>_{min})| = 2(<I_1><I_2>)^{1/2} / (<I_1> + <I_2>) \, | \gamma_{12}(\tau)|$

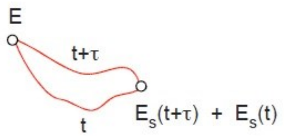
$<I_1> = <I_2> \Rightarrow \kappa = |\gamma_{12}(\tau)|$



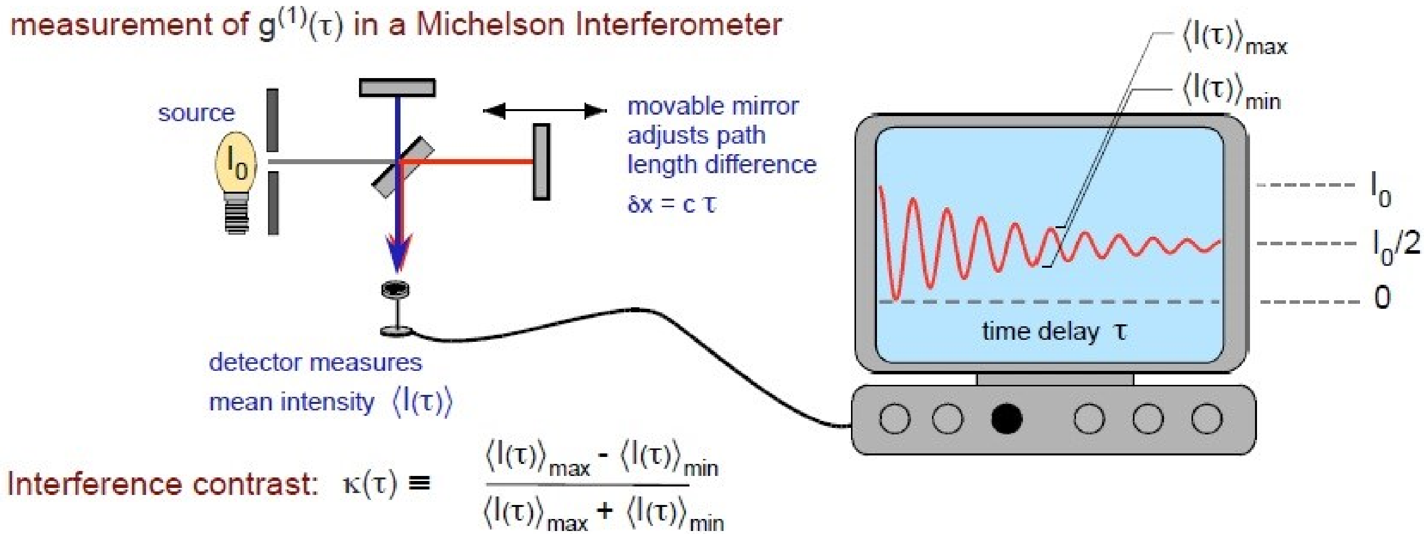
Definition:	$\lvert\gamma_{12}(\tau)\rvert = 1$	for all τ	$\Rightarrow$	complete coherence
	$0 < \lvert\gamma_{12}(\tau)\rvert < 1$	for some τ	$\Rightarrow$	partial coherence
	$\lvert\gamma_{12}(\tau)\rvert = 0$	for all τ	$\Rightarrow$	no coherence

- Normalized autocorrelation function: $g^{(1)}(\tau) \equiv \langle E(t+\tau)E^*(t)\rangle / \langle I\rangle$

  with $g^{(1)}(0)=1$ and $g^{(1)}(-\tau)=g^{(1)*}(\tau)$



Measurement of $g^{(1)}(\tau)$ in a Michelson Interferometer

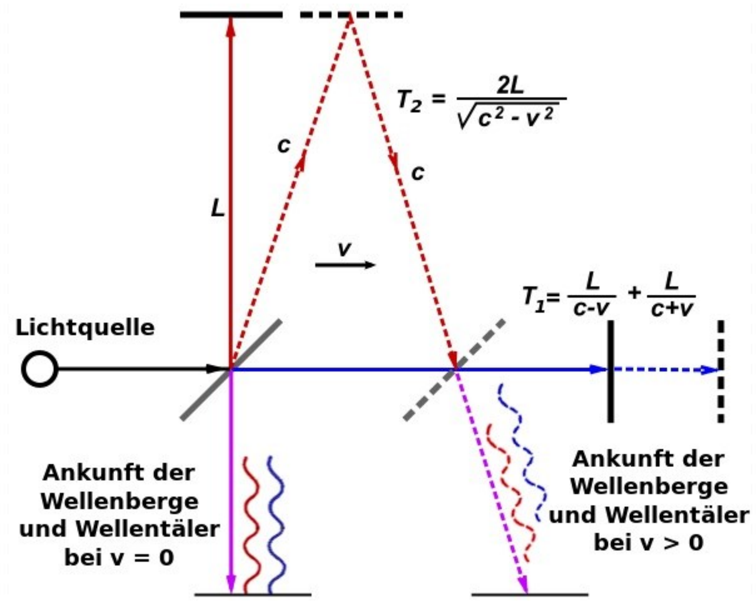


maximal coherence: Interference contrast maximal for all τ 

partial coherence: Interference contrast decreases for large τ 

The Michelson-Morley Experiment

by Albert Abraham Michelson: 1881 in Potsdam



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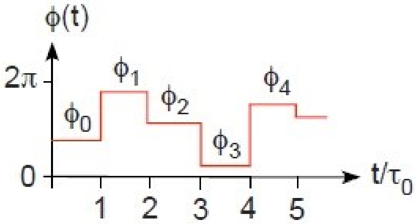
Normalized autocorrelation function: $g^{(1)}(\tau) \equiv \langle E(t+\tau)E^*(t)\rangle / \langle I\rangle$

with $g^{(1)}(0)=1$ and $g^{(1)}(-\tau)=g(1)^*(\tau)$

Example: sucessive wave trains of duration τ_0 and length $c\tau_0$

$E(t)= E_0\exp[i\omega t+i\phi(t)]$ with $\phi(t)$:

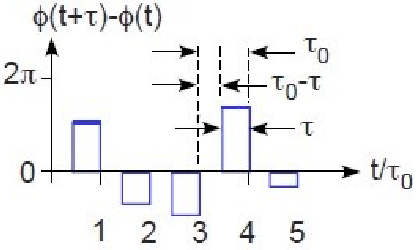


$\Rightarrow g^{(1)}(\tau) = e^{i\omega t} \langle e^{i(\phi(t+\tau) - \phi(t)}\rangle$

$0\leq\tau\leq\tau_0$:

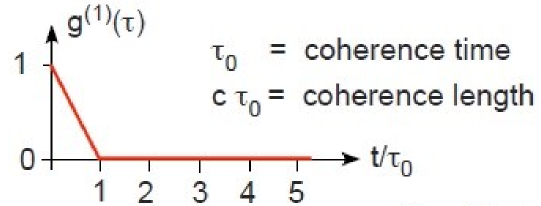
$\langle e^{i(\phi(t+\tau) - \phi(t)}\rangle = 1/N\tau_0 \sum_{n=0}^{N+1} \int_{n\tau_0}^{(n+1)\tau_0} dt\, e^{i(\phi(t+\tau) - \phi(t)}$

$= 1/N\tau_0 \sum_{n=0}^{N+1} \{(\tau_0-\tau) + \tau \exp(i\phi_{n+1} -\phi_n))\}$



$\Rightarrow g^{(1)}(\tau) = e^{\,i\omega\tau}(\tau_0-\tau)/\tau_0 \;\; x\;\; 0$ if $\tau_0 < \tau$

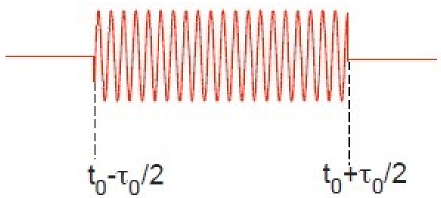
$x\;\; 1$ if $0 \leq\tau\leq\tau_0$



note: $\xi_l = \lambda/2\; \lambda/\Delta\lambda$

Coherence and emission spectrum:

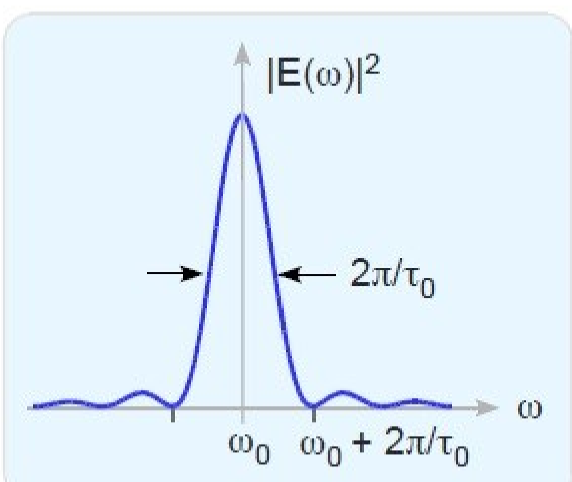
consider single wave train of duration τ_0, phase ϕ_0, frequency ω_0:



$E(t) = \exp[-i\omega_0 t - \phi_0]$ x 1 (if $t_0-\tau_0/2 \leq t \leq t_0+\tau_0/2$
x 0 otherwise

$$E(\omega)=1/\sqrt{2\pi}\int_{-\infty}^{\infty} dt\, E(t)\, e^{i\omega t} = \sqrt{2/\pi}(\sin(\omega-\omega_0)\tau_0/2)/(\omega-\omega_0) \bullet \exp(-i\phi_0)$$

N wave trains with the same frequency ω_0 but arbitrary phases ϕ_n, durations τ_n, starting times t_n:



$$E(\omega)=\sum_{n=1}^{N}\sqrt{2/\pi}\{\sin(\omega-\omega_0)\tau_n/2)/(\omega-\omega_0) \bullet \exp(i(\omega-\omega_0)t_n - i\phi_n)$$

$$|E(\omega)|^2 = \sum_{n=1}^{N} |E_n(\omega)|^2 =$$

$$= 2/\pi \sum_{n=1}^{N} \sin^2[(\omega-\omega_0)\tau_n/2)] / (\omega-\omega_0)^2$$

Emission bandwidth $\Delta\nu \cong 1/\tau$ with $\tau \equiv 1/N \sum_{n=1}^{N} \tau_n$

Example: Collision broadened light source

Molecules of a gas radiate light $E(t) = E_0 \exp[-i(\omega_0 t - \phi(t))]$ at frequency ω_0. Collisions yield random phase jumps, i.e., phase $\phi(t) \in [0, 2\pi]$ fluctuates.

Probability for a free flight of duration $t \in [\tau, \tau+d\tau]$: $P(t) = 1/\tau_0 \exp(-\tau/\tau_0)$

kinetic gas theory (τ_0 mean duration of free flight)

Coherence function: $g^{(1)}(\tau) = e^{i\omega_o\tau} < e^{\,i(\phi(t+\tau) - \phi(t))} >$

$e^{\,i(\phi(t+\tau) - \phi(t))}$ $= 1$ for free flights with duration $> \tau$

$= e^{i\chi}$ with $\chi \in [0, 2\pi]$ random for free flights with duration $< \tau$

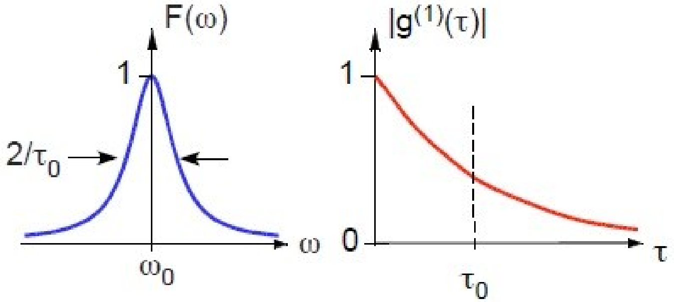
i.e., only flights of duration $t>\tau$ yield contribution to $< e^{\,i(\phi(t+\tau) - \phi(t))} >$:

$\Rightarrow$: $g^{(1)}(\tau) = e^{i\omega_o\tau} \int_\tau^\infty P(s)\, ds = e^{i\omega_o\tau} \exp(-\tau/\tau_0)$

$\Rightarrow$: $|g^{(1)}(\tau)| = \exp(-\tau/\tau_0)$

$\Rightarrow$: $F(\omega) = 1 / [1+(\omega-\omega_0)^2\tau_0^2$

(Wiener-Khintchine Theorem)



Wiener Khintchine Theorem:

$$E(\omega) \equiv \mathfrak{F}\,[E(t)] \equiv 1/\sqrt{(2\pi)}\ {}_{-\infty}\!\int^{\infty} dt\ E(t)\ e(i\omega t)$$

$$F(\omega) \equiv |\ E(\omega)\ |^2 \,/\ {}_{-\infty}\!\int^{\infty} dt\ \ |E(\omega)|^2$$

normalized spectral density

=> $$F(\omega) = 1/\sqrt{(2\pi)}\ \mathfrak{F}\,[g^{(1)}]\ ,$$

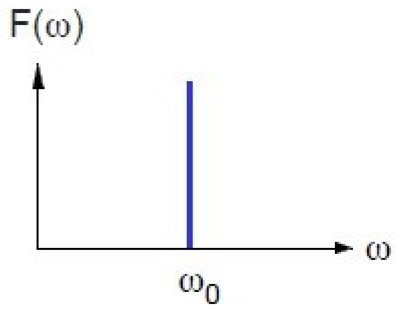
$\mathfrak{F} \equiv$ Fourier-Transform

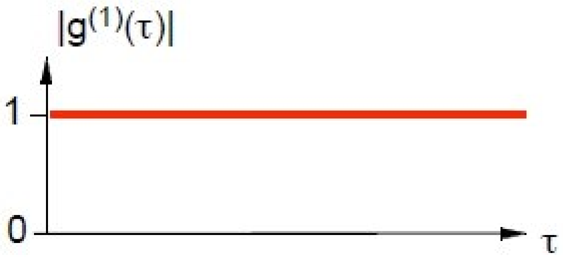
Example: monochromatic light

$E(t) = \exp[-i(\omega_0 t - \phi)]$

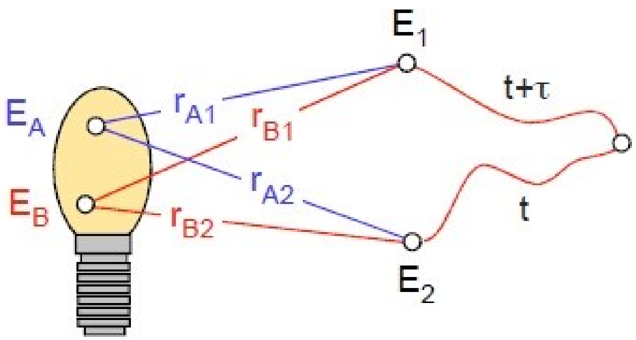
$g^{(1)}(\tau) = \exp(i\omega_0\tau)$

$|g^{(1)}(\tau)| = 1$





Spatial Coherence



Light Source: mutually incoherent point sources $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

$E_1 = E_{A1}+E_{B1}$ $\qquad E_{An}=E_A\exp(ir_{An}\omega/c)$

$E_2 = E_{A2}+E_{B2}$ $\qquad E_{Bn}=E_B\exp(ir_{Bn}\omega/c)$

$$<E_1(t+\tau)E_2^*(t)> = <E_{A1}(t+\tau)E_{A2}^*(t)> + <E_{B1}(t+\tau)E_{B2}^*(t)> + <E_{A1}(t+\tau)E_{B2}^*(t)> + <E_{B1}(t+\tau)E_{A2}^*(t)>$$

$$<I_n> = <E_n(t)E_n^*(t)> = <E_{An}(t)E_{An}^*(t)> + <E_{Bn}(t)E_{Bn}^*(t)> + <E_{An}(t)E_{Bn}^*(t)> + <E_{Bn}(t)E_{An}^*(t)>$$

$$\Rightarrow <I_1> = <I_2>$$

$$<E_{A1}(t+\tau)E_{A2}^*(t)> = <E_A(t+\tau)E_A^*(t)>\exp[i(r_{A1}-r_{A2})\omega/c] = <E_A(t+\tau_A)E_A^*(t)> \text{ with } \tau_A \equiv \tau+(r_{A1}-r_{A2})/c$$

$$<E_{B1}(t+\tau)E_{B2}^*(t)> = <E_B(t+\tau)E_B^*(t)>\exp[i(r_{B1}-r_{B2})\omega/c] = <E_B(t+\tau_B)E_B^*(t)> \text{ with } \tau_B \equiv \tau+(r_{B1}-r_{B2})/c$$

$$\Rightarrow <E_1(t+\tau)E_2^*(t)> = <E_A(t+\tau_A)E_A^*(t)> + E_B(t+\tau_B)E_B^*(t)>$$

$$\gamma_{12}(\tau) \equiv <E_1(t+\tau)E_2^*(t)> / (<I_1><I_2>)^{1/2} = ½[g^{(1)}(\tau_A) + g^{(1)}(\tau_B)] = 1/2\,[\exp(i\omega\tau_A-\tau_A/\tau_0) + \exp(i\omega\tau_B-\tau_B/\tau_0)]$$

$$4|\gamma_{12}(\tau)|^2 \equiv |g^{(1)}(\tau_A)|^2 + |g^{(1)}(\tau_B)|^2 + 2|g^{(1)}(\tau_A)|\,|g^{(1)}(\tau_B)| \cos(\omega(\tau_A-\tau_B))$$ interference term

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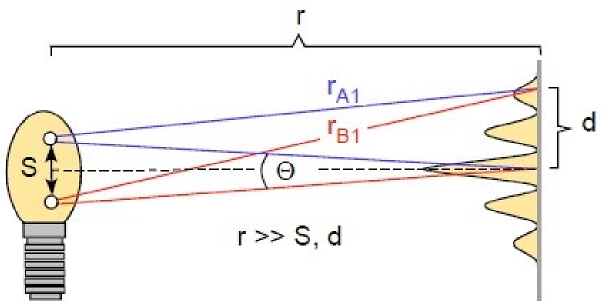
$$4|\gamma_{12}(\tau)|^2 \equiv |g^{(1)}(\tau_A)|^2 + |g^{(1)}(\tau_B)|^2 + 2|g^{(1)}(\tau_A)|\,|g^{(1)}(\tau_B)|\cos(\omega(\tau_A-\tau_B))$$

Interference term depends on $\tau_A-\tau_B = (r_{A1}-r_{A2})/c - (r_{B1}-r_{B2})/c$



Light Source: mutually incoherent point sources $g^{(1)}(\tau) = \exp(i\omega\tau - \tau/\tau_0)$

$r_{A2}-r_{B2} = 0 \Rightarrow \tau_A - \tau_B = (r_{A1}-r_{B1})/c$

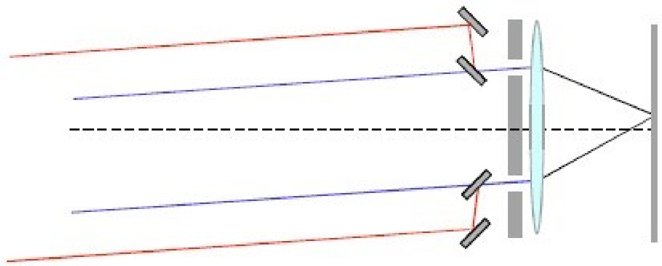
$r_{A1} \cong r + (d-S/2)^2/2r$, $r_{B1} \cong r + (d+S/2)^2/2r$

$\Rightarrow \tau_A - \tau_B \cong - Sd/2rc$

First minimum of $|\gamma_{12}(\tau)|^2$:

$\omega(\tau_A-\tau_B) = \pi$; $S \cong r\theta \Rightarrow d \cong \lambda/\theta$

transverse coherence length



Michelson stellar interferometer: adjustable slits, extension od slit separation by mirrors

Measurement of angular diameter of stars, angular separation of double stars, etc.