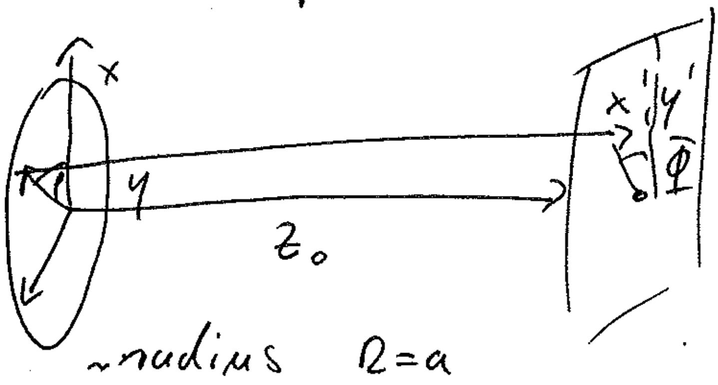
(a) $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$
(b) $\cos x \cos y = \frac{1}{2}(\cos(x-y) + \cos(x+y))$

Example 2

Diffraction from a circular Aperture



$x = \rho \cos \varphi$
$y = \rho \sin \varphi$

$x' = q \cos \Phi$
$y' = q \sin \Phi$

again our diffraction integral

$$E(x', y', z_0) = \frac{e^{-ikz_0}}{i\lambda z_0} \int_{\rho=0}^{a} \int_{\varphi=0}^{2\pi} e^{\frac{ik\rho q}{z_0} \cos(\varphi - \Phi)} \rho \, d\rho \, d\varphi$$

choose $\Phi = 0$ (does not matter)

$$E(x', y', z_0) = \frac{e^{-ikz_0}}{i\lambda z_0} \underbrace{\int_{\rho=0}^{a} \int_{\varphi=0}^{2\pi} e^{\frac{ik\rho \cdot q}{z_0} \cos \varphi} \rho \, d\rho \, d\varphi}_{\text{Bessel function of first kind}}$$

$$E(x', y', z_0) = \frac{e^{ikz_0}}{i\lambda z_0} 2\pi a^2 \cdot \frac{J_1(k \cdot a \cdot q / z_0)}{\frac{k \cdot a \cdot q}{z_0}}$$

$$I \simeq |E|^2 \sim \left(\frac{J_1\left(\frac{k \cdot a \cdot q}{z_0}\right)}{\frac{k \cdot a \cdot q}{z_0}} \right)^2$$

$\Rightarrow$ Airy disk   Sir George Biddel Airy
(1801–1892)

first zero for $J_1(u) = 0$ with $u = 3.83..$

$\Rightarrow \frac{k \cdot a \cdot q}{z_0} = 3.83$ or $\boxed{q_1 = 1.22 \, \frac{\lambda \cdot R}{2a}}$

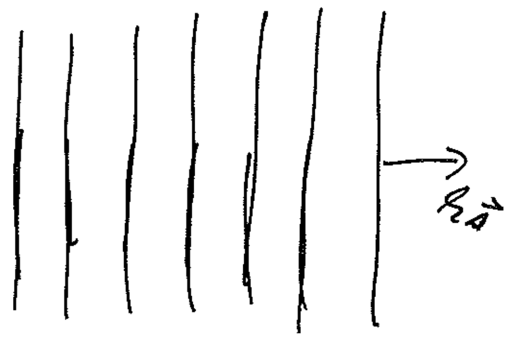
Coherence

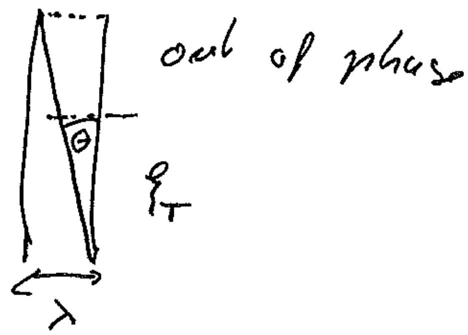
Important for Interference observation is that a fixed phase relation between the two waves exist on time scales $\tau \gg \frac{1}{\omega}$.

How to visualize the phase relations between two waves?

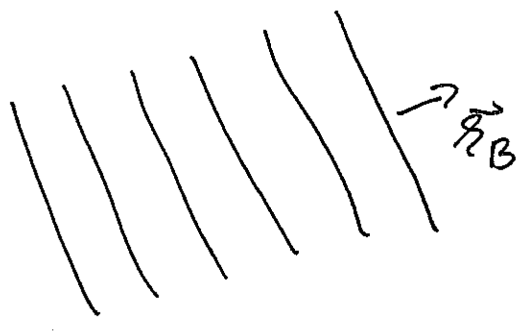
① transverse coherence

wave A





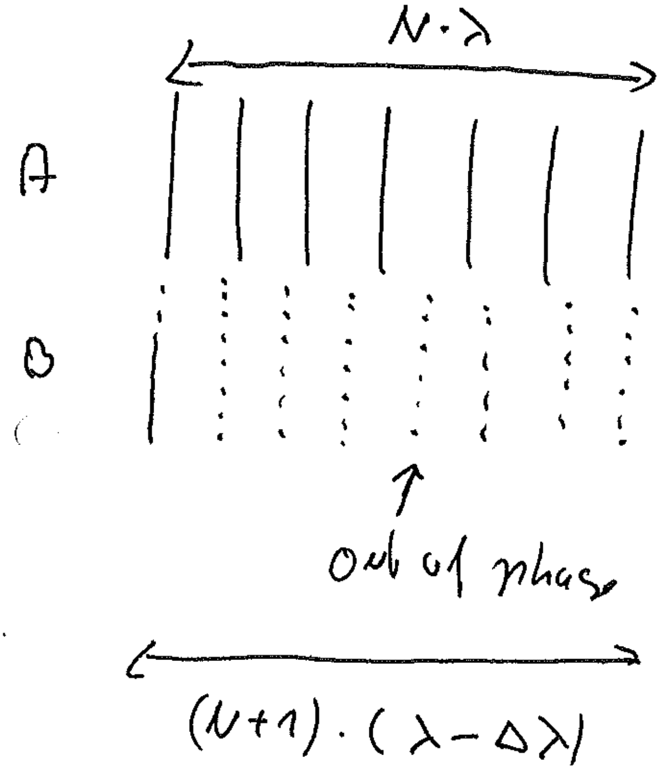
wave B



$$2\xi_T = \frac{\lambda}{\theta} = \frac{\lambda}{D} \cdot R$$

$$\boxed{\xi_T = \frac{\lambda \cdot R}{2 \cdot D}}$$

2 Longitudinal coherence length



wavelength λ

wavelength $\lambda - \Delta\lambda$

$$2\xi_L = N \cdot \lambda = (N+1)(\lambda - \Delta\lambda)$$

$$\Rightarrow \quad N \cdot \lambda = (N+1)(\lambda - \Delta\lambda) \qquad (N+1)\Delta\lambda = \lambda$$

$$N \approx \frac{\lambda}{\Delta\lambda}$$

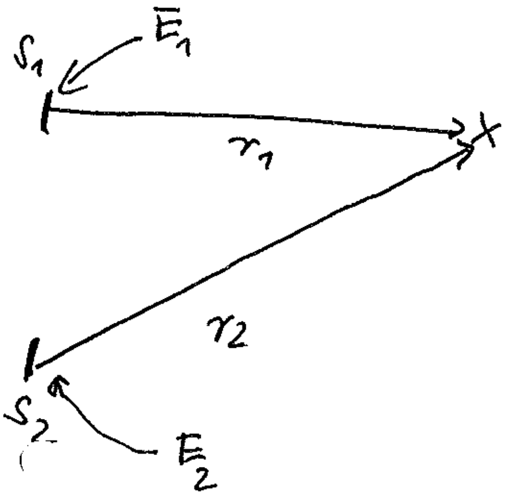
$$2\xi_L = \frac{\lambda^2}{\Delta\lambda} \Rightarrow \boxed{\xi_L = \frac{\lambda^2}{2\Delta\lambda}}$$

what are the coherence lengths telling us?

In order to observe interference phenomena the path length difference of two waves need to be smaller than the coherence lengths.

Formal development

Point P in space



field at position P

$$E_P(t) = \tilde{K}_1 E_1(t-t_1) + K_2 E_2(t-t_2)$$

$$t_1 = \frac{r_1}{c} \; ; \; t_2 = \frac{r_2}{c}$$

$E_1(t-t_1)$ field amplitude at S_1 to time $t-t_1$

K propagation factor (imaginary)

intensity at P at time t

$$I = \langle E_P(t) \, E_P^*(t) \rangle_T$$

$$=$$

$$I = K_1 K_1^* \langle E_1(t-t_1) E_1^*(t-t_1) \rangle_T + K_2 K_2^* \langle E_2(t-t_2) E_2^*(t-t_2) \rangle_T$$

$$+ K_1 K_2^* \langle E_1(t-t_1) E_2^*(t-t_2) \rangle_T + K_1^* K_2 \langle E_1^*(t-t_1) E_1(t-t_2) \rangle_T$$

first assumption: stationary fields, time average is independ of the origin of time

$$\langle E_1(t-t_1) E_1^*(t-t_1) \rangle_T = \langle E_1(t) E_1^*(t) \rangle_T$$

$$\langle E_1(t-t_1) E_2^*(t-t_2) \rangle_T = \langle E_1(t) E_2^*(t+\tau) \rangle_T$$

with $\tau = t_2 - t_1$

$$\Rightarrow I = I_1 + I_2 + 2 \, \mathrm{Re} \left[K_1 K_2^* \langle E_1(t+\tau) E_2^*(t) \rangle_T \right.$$

cross correlation function

$$\boxed{\text{mutual coherence function } \Gamma_{12}(\tau) = \langle E_1(t+\tau) E_2^*(t) \rangle_T}$$

$$I = I_1 + I_2 + 2 |K_1| |K_2| \, \mathrm{Re} \, \Gamma_{12}(\tau)$$

we want to normalize the mutual coherence function:

$$\Gamma_{11}(\tau) = \langle E_1(t+\tau) E_1^*(t) \rangle_T$$
$$\Gamma_{22}(\tau) = \langle E_2(t+\tau) E_2^*(t) \rangle_T$$

especially $I_1 = \Gamma_{11}(0) \, |K_1|^2$, $I_2 = \Gamma_{22}(0) \, |K_2|^2$

$$\Rightarrow \quad \gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)}\sqrt{\Gamma_{22}(0)}} = \frac{\langle E_1(t+\tau) E_2^*(t) \rangle_T}{\sqrt{\langle |E_1|^2 \rangle \langle |E_2|^2 \rangle}}$$

<u>complex degree of coherence</u>

finally $$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \, \mathrm{Re}\, \gamma_{12}(\tau)$$

$|\gamma_{12}(\tau)| = 1$ coherent

$|\gamma_{12}(\tau)| \to 0$ incoherent (not existing as an incoherent source would not radiate)

$0 < |\gamma_{12}(\tau)| < 1$ partial coherence

Visibility of fringes $V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = |\gamma_{12}(\tau)|$

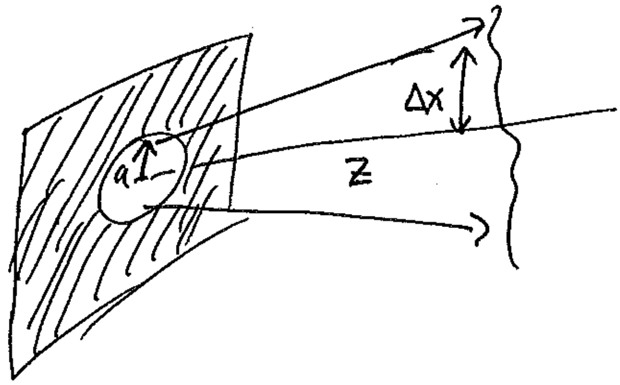
Calculating the degree of coherence

The van Cittert - Zernike Theorem for incoherent sources

$$\gamma(x_1, y_1; x_2, y_2) = \frac{e^{-i\psi} \iint I(\xi,\eta)\, e^{i\frac{2\pi}{\lambda z}(\Delta x \xi + \Delta y \eta)}\, d\xi\, d\eta}{\int_{-\infty}^{\infty} I(\xi,\eta)\, d\xi\, d\eta}$$

i.e. the complex degree of coherence is the Fourier Transform of the intensity distribution function!

Circular aperture radius a



$$\gamma(\Delta x, \Delta y) = e^{-i\psi} \left[2 \frac{J_1\left(\frac{2\pi a}{\lambda \cdot z} \sqrt{\Delta x^2 + \Delta y^2}\right)}{\frac{2\pi a}{\lambda \cdot z} \sqrt{(\Delta x)^2 + (\Delta y)^2}} \right]$$

See how similar this is to the intensity distribution of a coherently illuminated ~~of~~ circular aperture.

See previously: first zero for $J_1(u) = 0$ with $u = 3.83$

$$\boxed{\sqrt{\Delta x^2 + \Delta y^2} = \frac{1.22}{2} \cdot \frac{\lambda \cdot z}{a}}$$

coherence Area

$$\boxed{A_c = \frac{\lambda^2 z^2}{\pi a^2}}$$