June 26, 2001

Formulas for accelerator physics

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Curvature radius of ultra relativistic particles in a storage ring:

$$\frac{1}{r} = e \cdot \frac{B}{p} = \frac{c \cdot e \cdot B}{c \cdot p} = c \cdot e \cdot \frac{B}{E}$$

$$\frac{1}{r[m]} = 0.299792458 \cdot \frac{B[T]}{E[GeV]}; \quad 1[T] = 1[Vsec/m^2]$$

$(Example: \; r = 12.1849m; \; B = 1.221884T; \; E = 4.5GeV)$
$(Example: \; r = 191.7295m; \; B = 0.20877T; \; E = 12GeV)$
$(Example: \; r = 608m; \; B = 0.15142T; \; E = 27.6GeV)$

Quadrupole strength:

$$k = e \cdot \frac{g}{p} = \frac{c \cdot e \cdot g}{c \cdot p} = c \cdot e \cdot \frac{g}{E}$$

$$k[m^{-2}] = 0.299792458 \cdot \frac{g[T/m]}{E[GeV]}; \quad B = g \cdot x$$

$(Example: \; k = 0.6662m^{-2}; \; g = 10T/m; \; E = 4.5GeV)$

Sextupole strength:

$$m = e \cdot \frac{g'}{p} = \frac{c \cdot e \cdot g'}{c \cdot p} = c \cdot e \cdot \frac{g'}{E}$$

$$m[m^{-3}] = 0.299792458 \cdot \frac{g'[T/m^2]}{E[GeV]}; \quad B = \frac{1}{2} \cdot g' \cdot x^2$$

$(Example: \; m = 6.662m^{-3}; \; g' = 100T/m^2; \; E = 4.5GeV)$

Field index of a synchrotron magnet:

$$n = -\frac{r}{B_0} \cdot \frac{\Delta B_z}{\Delta x}$$

$$(Example:\ n = -22.26;\ r = 7.65;\ B_0 = 1.003T;\ \Delta B_z/\Delta x = 2.919T/m)$$

Focal length of a quadrupole:

$$\frac{1}{f} = \sqrt{k} \cdot \sin(\sqrt{k} \cdot l)$$

$$\approx k \cdot l \ \ for\ small\ k\ and\ l$$

$$(Example:\ f = 2.16m;\ k = 0.7m^{-2};\ l = 0.7m)$$

Wiggler parameter k:

$$k = \frac{e}{2\pi \cdot m_e \cdot c} \cdot B_{max} \cdot \lambda$$

$$k = 0.9337287 \cdot B_{max}[T] \cdot \lambda[cm]; \quad 1[kg] = 1[VAsec^3/m^2]$$

$$(Example:\ k = 7.40;\ B_{max} = 0.6T;\ \lambda = 13.2cm)$$
$$(Example:\ k = 28.0;\ B_{max} = 1.25T;\ \lambda = 24cm)$$
$$(Example:\ k = 1.61;\ B_{max} = 0.55T;\ \lambda = 3.14cm)$$

Focal length of a wiggler (vertical plane):

$$\frac{1}{f} = \frac{\tan\epsilon}{r} = \frac{2\pi^2 \cdot N \cdot k^2}{\lambda \cdot \gamma^2}; \quad \gamma = \frac{E}{E_0}$$

$$= \frac{1}{2}c^2 \cdot e^2 \cdot L \cdot \frac{B_{max}^2}{E^2}$$

$$\frac{1}{f[m]} = 0.04493776 \cdot L[m] \cdot \frac{B_{max}[T]^2}{E[GeV]^2}$$

$$(Example:\ f = 593m;\ L = 2.112m;\ B_{max} = 0.6T;\ E = 4.5GeV)$$
$$(Example:\ f = 113m;\ L = 4.00m;\ B_{max} = 1.0T;\ E = 4.5GeV)$$

Maximal electron deflection angle in a wiggler or undulator:

$$\Delta = \frac{k}{\gamma} = k \cdot \frac{E_0}{E} = \frac{c \cdot e}{2\pi} \cdot \lambda \cdot \frac{B_{max}}{E}$$

$$\Delta[rad] = 0.04771345 \cdot \lambda[m] \cdot \frac{B_{max}[T]}{E[GeV]} \ \ for\ electrons$$

$$(Example:\ \Delta = 3.18mrad;\ k = 28;\ E = 4.5GeV)$$
$$(Example:\ \Delta = 1.50mrad;\ k = 13.2;\ E = 4.5GeV)$$
$$(Example:\ \Delta = 3.18mrad;\ \lambda = 0.24m;\ B_{max} = 1.25T;\ E = 4.5GeV)$$

Number of ultra relativistic particles in a storage ring:

$$n = \frac{U \cdot I}{c \cdot Q}; \quad I = \frac{n \cdot Q}{T}; \quad c = \frac{U}{T}$$

$$(Example:\ n = 9.03 \cdot 10^{11};\ I = 0.15A;\ U = 289.2m;\ Q = e)$$
$$(Example:\ n = 2.88 \cdot 10^{12};\ I = 0.06A;\ U = 2304m;\ Q = e)$$

Velocity of a particle:

$$\frac{v}{c} = \sqrt{1 - \frac{E_0^2}{E^2}}$$

$$(Example:\ v/c = 0.999999994;\ E = 4.5GeV;\ electrons)$$
$$(Example:\ v/c = 0.992;\ E = 7.5GeV;\ protons)$$

Energy of a particle:

$$E = \sqrt{c^2 \cdot p^2 + E_0^2}$$

$$(Example:\ E = 3.7GeV;\ p = 3699.999965GeV/c;\ electrons)$$
$$(Example:\ E = 7.062603GeV;\ p = 7GeV/c;\ protons)$$

Kinetic energy of a particle beam:

$$E_{kin}[J] = 1.60217733 \cdot 10^{-19} \cdot E_{kin}[eV]$$

$$(Example:\ E_{total} = 2.89MJ;\ n = 2.2 \cdot 10^{13};\ E = 820GeV)$$

$$E_{kin}[J] = E_{kin}[m^2kg/sec^2]$$

$$(Example:\ E_{total} = 3.09MJ;\ n = 1;\ m = 2000kg;\ v = 200km/h)$$

α, β and γ function:

$$\gamma = \frac{1 + \alpha^2}{\beta}; \quad \alpha = -\frac{\beta'}{2}$$

β and α function at the end of a straight section of the length L:

$$\beta_L = \beta_0 - 2L \cdot \alpha_0 + L^2 \cdot \frac{1 + \alpha_0^2}{\beta_0}$$

$$\alpha_L = \alpha_0 - L \cdot \frac{1 + \alpha_0^2}{\beta_0}$$

Change of position and angle because of a kick in a storage ring:

$$dz = \frac{1}{2}\alpha_k\sqrt{\beta_k \cdot \beta_z} \cdot \frac{\cos(\pi Q_z - 2\pi|\varphi_z - \varphi_k|)}{\sin(\pi Q_z)}$$

$$dz' = -\frac{1}{2}\alpha_k \frac{\sqrt{\frac{\beta_k}{\beta_z}}}{\sin(\pi Q_z)} \cdot (\alpha_z \cdot \cos(\pi Q_z - 2\pi|\varphi_z - \varphi_k|) - \frac{\varphi_z - \varphi_k}{|\varphi_z - \varphi_k|} \cdot \sin(\pi Q_z - 2\pi|\varphi_z - \varphi_k|))$$

Change of position and angle because of a local bump kick:

$$dz = \alpha_k\sqrt{\beta_k \cdot \beta_z} \cdot \sin(\varphi_z - \varphi_k)$$

$$dz' = \alpha_k\sqrt{\frac{\beta_k}{\beta_z}} \cdot (\cos(\varphi_z - \varphi_k) - (-\frac{\beta_z'}{2}) \cdot \sin(\varphi_z - \varphi_k))$$

$(Example:\ dz = 2mm;\ \alpha_k = 0.1mrad;\ \beta_k = \beta_z = 20m/rad;$
$\varphi_z - \varphi_k = \pi/2)$

Orbit lengthening because of a horizontal kick in a storage ring:

$$dl = D_x \cdot \alpha_k$$

Enlargement of the quadrupole strength in F(D) quadrupoles leads there to a reduction of the β_x(β_z) function

Tune shift due to a change of the quadrupole strength:

$$\Delta Q = \frac{1}{4\pi} \cdot \beta \cdot L \cdot \Delta k$$

$(Example:\ \Delta Q = 0.1114;\ \beta = 20m;\ L = 0.7m;\ \Delta k = 0.1m^{-2})$

Vertical tune shift due to a wiggler:

$$\Delta Q_z = \frac{\pi \cdot \beta_z \cdot N \cdot k^2}{2 \cdot \gamma^2 \cdot \lambda} = \frac{c^2 \cdot e^2 \cdot \beta_z \cdot L \cdot B_{max}^2}{8\pi \cdot E^2}$$

$$\Delta Q_z = 0.003576033 \cdot \beta_z[m] \cdot L[m] \cdot \frac{B_{max}[T]^2}{E[GeV]^2}$$

$(Example:\ \Delta Q_z = 0.0017;\ \beta_z = 13m;\ L = 2.112m;\ B_{max} = 0.60T;$
$E = 4.5GeV)$

$(Example:\ \Delta Q_z = 0.0106;\ \beta_z = 16m;\ L = 2.400m;\ B_{max} = 1.25T;\ E = 4.5GeV)$
$(Example:\ \Delta Q_z = 0.0170;\ \beta_z = 24m;\ L = 4.000m;\ B_{max} = 1.00T;\ E = 4.5GeV)$

Fraction of the tune of a storage ring:

$$frac(Q) = \frac{1}{2\pi}\arccos(\frac{1}{2}trace(M)) = \frac{1}{2\pi}\arccos(\frac{1}{2}(M11 + M22))$$

Tune of a storage ring:

$$Q = \frac{1}{2\pi}\cdot\int\frac{1}{\beta}ds$$

Chromaticity of a storage ring:

$$\frac{\Delta Q}{\frac{\Delta E}{E}} = \frac{1}{4\pi}\int(k - m\cdot D)\cdot\beta\cdot dl$$

$$(k < 0 : F - quadrupole, m < 0 : F - sextupole)$$

Momentum compaction factor of a storage ring:

$$\alpha_c = \frac{\frac{\Delta L}{L}}{\frac{\Delta p}{p}} = \frac{\frac{\Delta T}{T}}{\frac{\Delta E}{E}}\ (ultra\ relativistic)$$

$$\alpha_c = \frac{\frac{\Delta T}{T}}{\frac{\Delta p}{p}} + \frac{1}{\gamma^2}\ (relativistic)$$

Transition energy:

$$\gamma_{tran} = \frac{E}{E_0} = \frac{1}{\sqrt{\alpha_c}}$$

$(Example:\ E = 3.9MeV;\ electrons;\ \alpha_c = 0.017)$
$(Example:\ E = 8.77GeV;\ protons;\ \alpha_c = 0.01145)$

Emittance after adiabatic damping in a storage ring:

$$\epsilon_x \sim \frac{1}{\beta\cdot\gamma}\ (relativistic)$$

$$\epsilon_x \sim \frac{1}{\gamma} = \frac{E_0}{E}\ (ultra\ relativistic)$$

Emittance of a radiation damped beam in a storage ring:

$$\epsilon_x \sim E^2$$

Emittance dependance in a storage ring:

$$\Delta\epsilon_x \sim \frac{D_x^2}{\beta_x \cdot abs(r)},\ if\ D_x' = 0,\ \alpha_x = 0\ and\ D_x << abs(r)$$

Beam dimensions in an electron storage ring:

$$\sigma_x = \sqrt{\epsilon_x \cdot \beta_x + (D_x \cdot \sigma_e)^2}$$

$$\sigma_z = \sqrt{\epsilon_z \cdot \beta_z}$$

$$\sigma_x' = \sqrt{\epsilon_x \frac{1+\alpha_x^2}{\beta_x} + (D_x' \cdot \sigma_e)^2}$$

$$\sigma_z' = \sqrt{\epsilon_z \frac{1+\alpha_z^2}{\beta_z}}$$

$(Example:\ \sigma_x = 3.0mm;\ \epsilon_x = 0.4mm \cdot mrad;\ \beta_x = 20m/rad;$
$D_x = 1m;\ \sigma_e = 0.001)$
$(Example:\ \sigma_z' = 28\mu rad;\ \epsilon_z = 0.008mm \cdot mrad;\ \beta_z = 10m/rad;\ \alpha_z = 0)$

Touschek scattering means elastic electron electron scattering within a bunch

Beam lifetime due to inelastic electron scattering with residual gas nucleons (bremsstrahlung production):

$$\tau_B = \frac{1}{\alpha \cdot c \cdot \frac{(4r_e)^2}{3} \cdot Z(Z+1) \cdot \ln(183 \cdot Z^{-1/3}) \cdot (-\ln\epsilon - \frac{5}{8}) \cdot N_g}$$

$$\tau_B[sec] = \frac{1.079323 \cdot 10^{22}}{Z(Z+1) \cdot \ln(183 \cdot Z^{-1/3}) \cdot (-\ln\epsilon - \frac{5}{8}) \cdot N_g[m^{-3}]}$$

$$N_g[m^{-3}] = \frac{\rho[kg/m^3]}{2Z \cdot m_p[kg]} \cdot \frac{p[mbar]}{1013}$$

$$\tau_B[sec] = \frac{8.019427 \cdot 10^{-4}}{(-\ln\epsilon - \frac{5}{8}) \cdot p[mbar]} \quad for\ Z = 7$$

$(Example:\ \tau_B = 74718sec = 20.75h;\ Z = 7(nitrogen);\ \epsilon = 0.0025;$
$N_g = 1.053912 \cdot 10^{14} m^{-3};\ \rho = 1.25kg/m^3;\ p = 2 \cdot 10^{-9} mbar)$

Beam lifetime due to elastic electron scattering with residual gas nucleons:

$$\tau_{el} = \frac{2E^2}{\pi \cdot c \cdot E_0^2 \cdot (2r_e \cdot Z)^2 \cdot (\frac{<\beta_x>}{A_x} + \frac{<\beta_z>}{A_z}) \cdot N_g}$$

$$\tau_{el}[sec] = \frac{2.560327 \cdot 10^{26} \cdot E[GeV]^2}{Z^2 \cdot (\frac{<\beta_x>}{A_x} + \frac{<\beta_z>}{A_z}) \cdot N_g[m^{-3}]}$$

$$N_g[m^{-3}] = \frac{\rho[kg/m^3]}{2Z \cdot m_p[kg]} \cdot \frac{p[mbar]}{1013}$$

$$\tau_{el}[sec] = \frac{99.1573 \cdot E[GeV]^2}{(\frac{<\beta_x>}{A_x} + \frac{<\beta_z>}{A_z}) \cdot p[mbar]} \quad for\ Z = 7$$

$(Example:\ \tau_{el} = 210133sec = 58.37h;\ Z = 7(nitrogen);\ E = 4.5GeV;$
$\beta_x = 20m;\ A_x = 60 \cdot 10^{-6} m * rad;\ \beta_z = 20m;\ A_z = 4.5 \cdot 10^{-6} m * rad;$
$N_g = 1.053912 \cdot 10^{14} m^{-3};\ \rho = 1.25kg/m^3;\ p = 2 \cdot 10^{-9} mbar)$

$(Example:\ \tau_{el} = 84992sec = 23.61h;\ Z = 7(nitrogen);\ E = 6GeV;$
$\beta_x = 30m;\ A_x = 30 \cdot 10^{-6} m * rad;\ \beta_z = 30m;\ A_z = 1.5 \cdot 10^{-6} m * rad;$
$N_g = 1.053912 \cdot 10^{14} m^{-3};\ \rho = 1.25kg/m^3;\ p = 2 \cdot 10^{-9} mbar)$

Cavity power:

$$P = \frac{U^2}{2R_s}$$

$(Example:\ P = 110kW;\ U = 1.755MV;\ R_s = 14M\Omega)$

Dispersion in a cavity leads to a displacement of the closed orbit during acceleration or deacceleration and therefore to an excitation of radial betatron oscillations

Polarization time in a storage ring (Sokolov-Ternov):

$$P_{max} = \frac{8}{5\sqrt{3}} = 0.9237604$$

$$\tau = \frac{8}{5\sqrt{3}} \cdot \frac{E_0^6 \cdot R \cdot r^2}{\hbar^2 \cdot c^2 \cdot r_e \cdot E^5}$$

$$\tau[sec] = 98.65992 \cdot \frac{R[m] \cdot r[m]^2}{E[GeV]^5} \ \ for\ electrons$$

$(Example:\ \tau = 365sec;\ R = 46.0276m;\ r = 12.1849m;\ E = 4.5GeV)$
$(Example:\ \tau = 1513sec;\ R = 1008m;\ r = 608m;\ E = 30GeV)$

Luminosity:

$$L = \frac{1}{4\pi \cdot e^2 \cdot f_u} \cdot \frac{I_1 \cdot I_2}{\sigma_x \cdot \sigma_z}$$

$(Example:\ L = 2.49 \cdot 10^{28} cm^{-2} sec^{-1};\ I_1 = 1mA;\ I_2 = 1mA;$
$f_u = 1.037MHz;\ \sigma_x = 0.075cm;\ \sigma_z = 0.0016cm;\ (\kappa = 1\%))$

Vertical tune shift at one interaction point (IP) in a storage ring (tune splitting; only the higher tune moves):

$$\Delta Q_z = \frac{r_e \cdot E_0 \cdot N_{bunch} \cdot \beta_z(IP)}{2\pi \cdot E \cdot \epsilon_x \cdot (\sqrt{\kappa \cdot \beta_x(IP) \cdot \beta_z(IP)} + \kappa \cdot \beta_z(IP))}$$

$$\Delta Q_z \approx \frac{r_e \cdot E_0 \cdot N_{bunch}}{2\pi \cdot E \cdot \sqrt{\epsilon_x \cdot \epsilon_z}} \cdot \sqrt{\frac{\beta_z(IP)}{\beta_x(IP)}}$$

$(Example:\ \Delta Q_z = 0.022;\ N_{bunch} = 2.7 \cdot 10^{11};\ \beta_z(IP) = 4cm;$
$E = 5.3GeV;\ \epsilon_x = 0.56 \cdot 10^{-6} m;\ \kappa = .05;\ \beta_x(IP) = .63m)$

Synchrotron frequency:

$$f_s = f_u \cdot \sqrt{\frac{H \cdot \alpha_c \cdot e \cdot U_{HF} \cdot \cos\psi_s}{2\pi \cdot E}}$$

$(Example:\ f_s = 39.9kHz;\ f_u = 1.037MHz;\ H = 482;\ \alpha_c = 0.016;$
$U_{HF} = 6MV;\ \psi_s = 20°;\ E = 4.5GeV)$