

# Methoden moderner Röntgenphysik: Streuung und Abbildung

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Lecture 4	Vorlesung zum Haupt- oder Masterstudiengang Physik, SoSe 2021 G. Grübel, O. Seeck, V. Markmann, F. Lehmkuhler, Andre Philippi-Kobs, M. Martins		
Location	online		
Date	Tuesdays	12:30 - 14:00	(starting 6.4.)
	Thursdays	8:30 - 10:00	(until 8.7.)



# Methoden moderner Röntgenphysik: Online Info

Tuesday Zoom-Meeting

<https://desy.zoom.us/j/92674682486>

Meeting ID: 926 7468 2486

Passcode: 144456

Thursday Zoom-Meeting

<https://desy.zoom.us/j/99738625981>

Meeting ID: 997 3862 5981

Passcode: 841881

Tutorial Zoom-Meeting

<https://desy.zoom.us/j/95288979489>

Meeting ID: 952 8897 9489

Passcode: 832350



# Literature

## Basic concepts:

### Elements of Modern X-Ray Physics

J. A. Nielsen and D. McMorrow, J. Wiley&Sons (2001)

### X-Ray Diffraction

B.E. Warren, DOVER Publications Inc., New York

### Principles of Optics

M. Born and E. Wolf, Cambridge University Press, 7<sup>th</sup> ed.

### Soft X-rays and Extreme Ultraviolet Radiation

D. Attwood, Cambridge University Press (2000)

<http://www.coe.berkeley.edu/AST/sxreuv/>

### Physik der Teilchenbeschleuniger und Synchrotronstrahlungsquellen

K. Wille, Teubner Studienbücher 1996

# Lecture Notes

[https://photon-science.desy.de/research/research\\_teams/coherent\\_x\\_ray\\_scattering/teaching/index\\_eng.html](https://photon-science.desy.de/research/research_teams/coherent_x_ray_scattering/teaching/index_eng.html)



# Methoden moderner Röntgenphysik: Streuung und Abbildung

## Part I:

### Basics of X-ray Physics

by Gerhard Grübel (GG)

#### Introduction

Overview, Introduction to X-ray Scattering

#### X-ray Scattering Primer & Sources of X-rays +Synchrotron Radiation

Elements of X-ray Scattering, Laboratory Sources, Accelerator Bases Sources

#### Reflection and Refraction from Interfaces

Snell's Law, Fresnel Equations

#### Kinematical Diffraction (I)

Diffraction from an Atom, a Molecule, from Liquids, Glasses, ...



#### Kinematical Diffraction (II)

Diffraction from a Crystal, Reciprocal Lattice, Structure Factor, ...

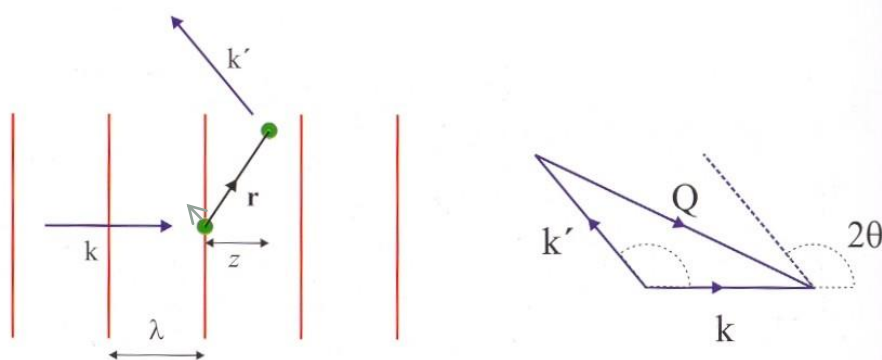
# Kinematical Diffraction

# Kinematical Diffraction

One of the main applications of X-rays is the determination of structure(s) using diffraction.

Assume the scattering to be weak; multiple scattering effects are to be neglected: weak scattering limit  $\equiv$  kinematical approximation.

Consider a 2 electron system:



$$z = r \cos \alpha; \quad k z = k r \cos \alpha = \mathbf{k} \cdot \mathbf{r}$$

$$y = r \cos \beta; \quad k' y = k' r \cos \beta = \mathbf{k}' \cdot \mathbf{r}$$

Path- or phase difference:

$$\Delta \Phi = z - y = \mathbf{k} \cdot \mathbf{r} - \mathbf{k}' \cdot \mathbf{r} = \mathbf{Q} \cdot \mathbf{r}$$

with

$$Q = \left(\frac{4\pi}{\lambda}\right) \sin \theta$$

Elastic scattering

$$k = k'$$

Scattering amplitude for 2 electrons:

$$A(\mathbf{Q}) = -r_0 [1 + e^{i\mathbf{Q}\mathbf{r}}]$$

$$I(\mathbf{Q}) = A(\mathbf{Q})A(\mathbf{Q})^*$$

$$= 2r_0^2 [1 + \cos(\mathbf{Q}\mathbf{r})]$$

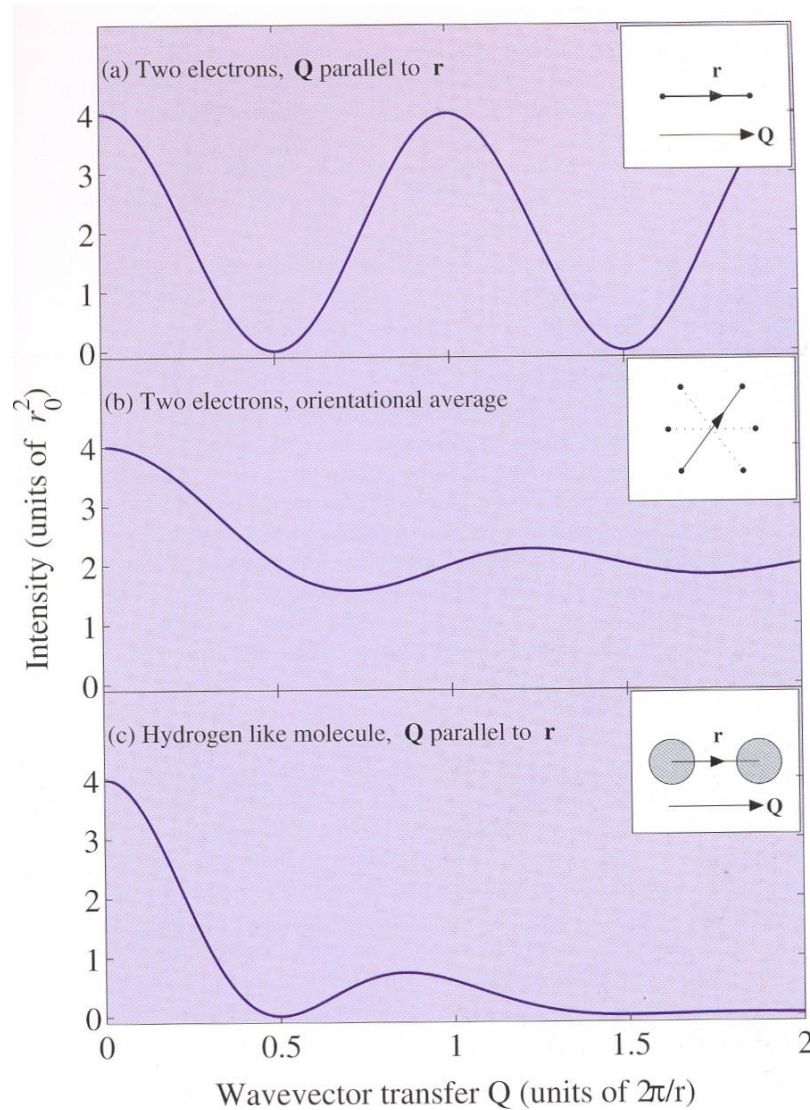
see Fig. 4.2

For many electrons:

$$A(\mathbf{Q}) = -r_0 \sum_{j'} e^{i\mathbf{Q}\mathbf{r}_{j'}}$$



# Kinematical Diffraction

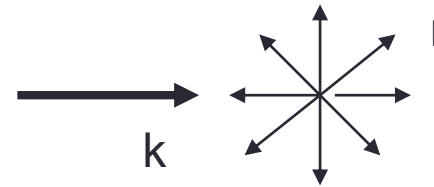


Two electron system:

$$I(Q) = 2r_0^2 [1 + \cos(Qr)]$$

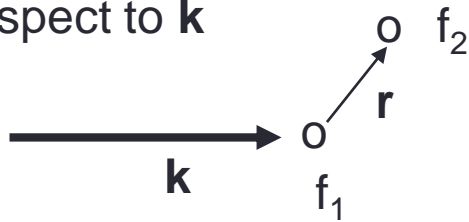
$Q \parallel r$

orientational average



“smeared”, no more “point-like” particles

For many systems, e.g. molecules, the orientation of  $\mathbf{r}$  will be random with respect to  $\mathbf{k}$



Orientational averaging:

Assume one electron at  $r=0$ , a second at  $r$

$$A(\mathbf{Q}) = f_1 + f_2 e^{i\mathbf{Q}\mathbf{r}}$$

$$I(\mathbf{Q}) = f_1^2 + f_2^2 + f_1 f_2 e^{i\mathbf{Q}\mathbf{r}} + f_1 f_2 e^{-i\mathbf{Q}\mathbf{r}}$$

Orientational averaging:

$$\langle e^{i\mathbf{Q}\mathbf{r}} \rangle = \langle e^{-i\mathbf{Q}\mathbf{r}} \rangle$$

$$\langle I(\mathbf{Q}) \rangle = f_1^2 + f_2^2 + 2f_1 f_2 \langle e^{i\mathbf{Q}\mathbf{r}} \rangle$$

$$\langle e^{i\mathbf{Q}\mathbf{r}} \rangle = \frac{\int e^{i\mathbf{Q}\mathbf{r} \cos\theta} \sin\theta \, d\theta}{\int \sin\theta \, d\theta \, d\phi} \leftarrow 4\pi$$

$$\int \exp(i\mathbf{Q}\mathbf{r} \cos\theta) \sin\theta \, d\theta \, d\Phi$$

$$= 2\pi \int e^{i\mathbf{Q}\mathbf{r} \cos\theta} \sin\theta \, d\theta$$

$$= 2\pi \left(-\frac{1}{i\mathbf{Q}\mathbf{r}}\right) \int_{i\mathbf{Q}\mathbf{r}}^{-i\mathbf{Q}\mathbf{r}} e^x \, dx$$

$$= 4\pi \frac{\sin(\mathbf{Q}\mathbf{r})}{\mathbf{Q}\mathbf{r}}$$

$$\langle I(\mathbf{Q}) \rangle = f_1^2 + f_2^2 + 4\pi f_1 f_2 \frac{\sin(\mathbf{Q}\mathbf{r})}{\mathbf{Q}\mathbf{r}}$$

see figure 4.2 b

if the position of the electrons distributed or smeared: see Figure 4.2c

Orientational averaging  
 $\Rightarrow$  isotropic scattering signal  
 $I(Q)$  instead of  $I(\mathbf{Q})$





# Scattering From an Atom

Scattering amplitude of an atom  $\equiv$  atomic form factor  $f_0(\mathbf{Q})$  [in units of  $r_0$ ]

$\rho(\mathbf{r})$ : electronic number density  $\equiv$  charge density

$$f_0(\mathbf{Q}) = \int \rho(\mathbf{r}) e^{i\mathbf{Q}\mathbf{r}} d\mathbf{r} = \begin{cases} Z & \text{for } \mathbf{Q} \rightarrow 0 \\ 0 & \text{for } \mathbf{Q} \rightarrow \infty \end{cases}$$

	$a_1$	$b_1$	$a_2$	$b_2$	$a_3$	$b_3$	$a_4$	$b_4$	$c$
C	2.3100	20.8439	1.0200	10.2075	1.5886	0.5687	0.8650	51.6512	0.2156
O	3.0485	13.2771	2.2868	5.7011	1.5463	0.3239	0.8670	32.9089	0.2508
F	3.5392	10.2825	2.6412	4.2944	1.5170	0.2615	1.0243	26.1476	0.2776
Si	6.2915	2.4386	3.0353	32.333	1.9891	0.6785	1.5410	81.6937	1.1407
Cu	13.338	3.5828	7.1676	0.2470	5.6158	11.3966	1.6735	64.820	1.5910
Ge	16.0816	2.8509	6.3747	0.2516	3.7068	11.4468	3.683	54.7625	2.1313
Mo	3.7025	0.2772	17.236	1.0958	12.8876	11.004	3.7429	61.6584	4.3875

Table 4.1: J. Als-Nielsen & D. McMorrow

**Note:** atomic form factor is FT of electronic charge distribution

$f_0\left(\frac{Q}{4\pi}\right)$  tabulated:

$$f_0\left(\frac{Q}{4\pi}\right) = \sum_{j=1}^4 a_j e^{-b_j \left(\frac{Q}{4\pi}\right)^2} + c$$

**Note:**

$$f = f_0(Q) + f' + f''$$

corrections  $f'$  and  $f''$  arise from the fact that the electrons are bound in the atom



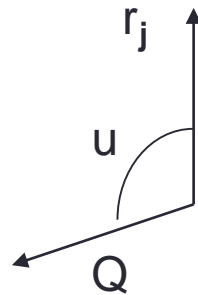
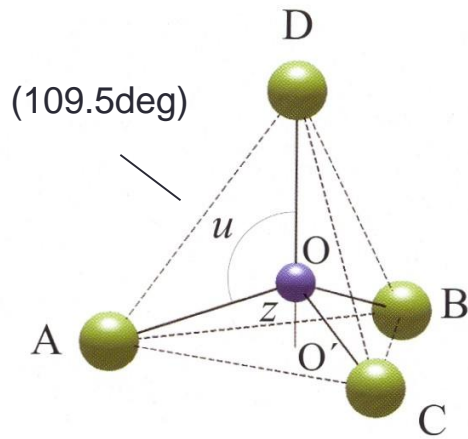
# Scattering from a Molecule

$$F^{\text{mol}}(\mathbf{Q}) = \sum_{r_j} f_j(\mathbf{Q}) e^{i\mathbf{Q}r_j}$$

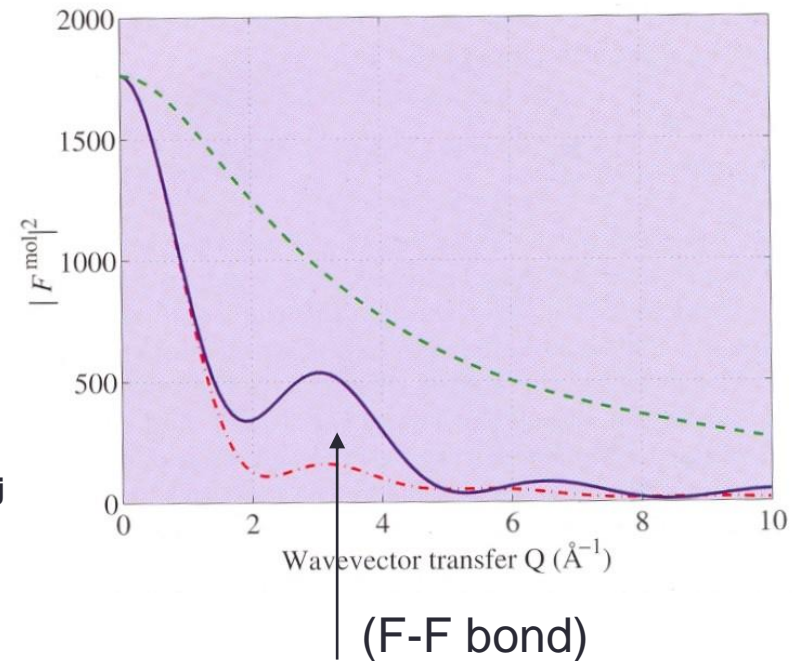
- CF<sub>4</sub>
- · - · - CF<sub>4</sub>    Q not || C-F
- - - - molybdenum  
(also 42 electrons)

## Example: CF<sub>4</sub> (tetraeder)

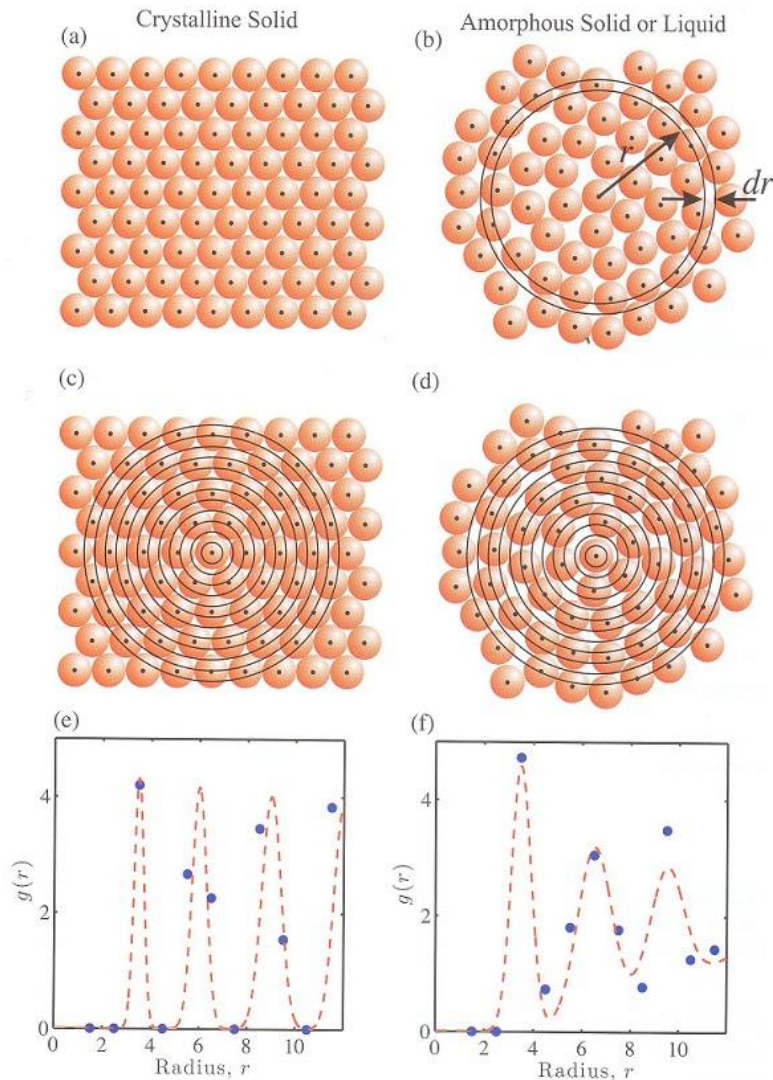
assume OA = OB = OC = OD = 1; z = OO' = cos(u) =  $\frac{1}{3}$



$$Qr_j = Qr_j \cos(u) = \left(\frac{1}{3}\right)Qr_j$$



# Scattering From Liquids and Glasses



The positions of atoms in non-crystalline materials change over a wide range of timescales (from nanoseconds in the case of liquids to millennia or more in the case of glasses).

X-rays are a fast probe delivering snapshots of the structure as shown schematically in the figure.

Radial density: 
$$\rho(r) = \frac{N(r)}{2\pi r dr}$$

with  $N(r)$  number of atoms in annulus  $r+dr$ ,  
with  $2\pi r dr$  being the annulus' area

Radial distribution function: 
$$g(r) = \frac{\rho(r)}{\rho_{at}}$$

$\rho_a$  average areal number density

The radial distribution function of a non-crystalline material damps and broadens as a function of  $r$  with  $g(r)$  tending to unity.



# The Liquid Structure Factor

Consider mono-atomic or mono-molecular systems:

$$I(\mathbf{Q}) = f(\mathbf{Q})^2 \sum_n e^{i\mathbf{Q}\mathbf{r}_n} \sum_m e^{i\mathbf{Q}\mathbf{r}_m} = f(\mathbf{Q})^2 \sum_n \sum_m e^{i\mathbf{Q}(\mathbf{r}_n - \mathbf{r}_m)}$$

with  $f(\mathbf{Q})$  form factor

separate summations

$$I(\mathbf{Q}) = Nf(\mathbf{Q})^2 + f(\mathbf{Q})^2 \sum_n \sum_{m \neq n} e^{i\mathbf{Q}(\mathbf{r}_n - \mathbf{r}_m)}$$

Replace  $m \neq n$  sum by integral and separate out average density  $\rho_{at}$ :

$$I(\mathbf{Q}) = \underbrace{Nf(\mathbf{Q})^2 + f(\mathbf{Q})^2 \sum_n \int_V [\rho_n(\mathbf{r}_{nm}) - \rho_{at}] e^{i\mathbf{Q}(\mathbf{r}_n - \mathbf{r}_m) dV_m}}_{I_{SRO}(\mathbf{Q})} + \underbrace{f(\mathbf{Q})^2 \rho_{at} \sum_n \int_V e^{i\mathbf{Q}(\mathbf{r}_n - \mathbf{r}_m) dV_m}_{I_{SAXS}(\mathbf{Q})}}$$

$I_{SRO}(\mathbf{Q})$

measures short-range order (SRO) since

$\rho_n(\mathbf{r}_{nm}) \rightarrow \rho_{at}$  after few atomic spacings

and the term oscillates then towards zero

$I_{SAXS}(\mathbf{Q})$

contributes only for  $Q \rightarrow 0$

(otherwise oscillates to zero)

where  $\rho_n(\mathbf{r}_{nm}) dV_m$  is the number of atoms in element  $dV_m$  located at  $\mathbf{r}_m - \mathbf{r}_n$  relative to  $\mathbf{r}_n$ .



# The Liquid Structure Factor

Average over different choices for the origin  $\langle \rho_n(\mathbf{r}_{nm}) \rangle \rightarrow \rho(r)$

Isotropy:  $\rho(\mathbf{r}) = \rho(r)$ : 
$$I(Q) = Nf(Q)^2 + Nf(Q)^2 \int_0^\infty [\rho(r) - \rho_{at}] 4\pi r^2 \left(\frac{\sin(Qr)}{Qr}\right) dr$$

$$\Rightarrow S(Q) = \frac{I_{SRO}(Q)}{Nf(Q)^2} = 1 + \frac{4\pi}{Q} \int_0^\infty r [\rho(r) - \rho_{at}] \sin(Qr) dr$$

For  $Q \rightarrow \infty$ :  $S(Q) \rightarrow 1$

For  $Q \rightarrow 0$ :  $\frac{\sin(Qr)}{Q} \rightarrow r \Rightarrow S(Q) = 1 + 4\pi \int_0^\infty r^2 dr [\rho(r) - \rho_{at}]$

$S(Q) \propto [\rho(r) - \rho_{at}]$  proportional to the density fluctuations in the system

Density fluctuations become strong when the compressibility  $\kappa_T$  is large. At a critical point, the compressibility diverges and  $S(0)$  might become so strong that the sample becomes opaque: critical opalescence.

For an ideal gas:  $P = \rho_{at} k_B T$  and  $\kappa_T = \frac{1}{(\rho_{at} k_B T)}$  since  $\kappa_T = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial P}\right)_T$

For interacting gas:  $\kappa_T$  diverges at critical point

actually:  $S(0) = \rho_{at} \kappa_T k_B T$  (= 1 at all  $Q$  for an ideal gas)



# The Pair Correlation Function $g(r)$ :

$$S(Q) = 1 + \frac{4\pi}{Q} \int_0^\infty r [\rho(r) - \rho_{\text{at}}] \sin(Qr) dr$$

⇒ Radial distribution function for a liquid or glass:

$$g(r) = 1 + \left( \frac{1}{2\pi^2 r \rho_{\text{at}}} \right) \int_0^\infty Q [S(Q) - 1] \sin(Qr) dQ$$

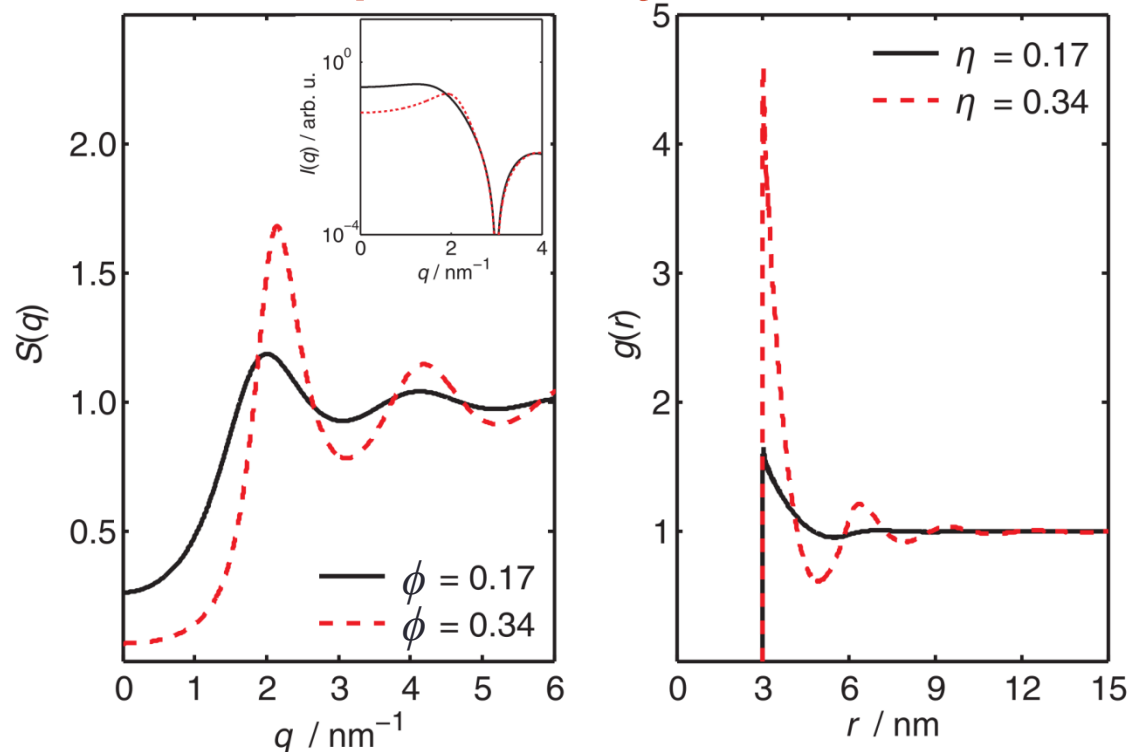
Note:

It was assumed that the scattering is dominated by elastic scattering events. This holds for a glass but not for a liquid. However e.g. 10 keV photons are far more energetic than any relevant excitation in a liquid (typically 10 meV); thus any inelastic effect is negligibly small.

However this does not hold for neutrons.



# Example: Hard sphere system



- Interaction only via steric repulsion
- model system for liquids, colloids, glasses
- only dependent on volume fraction  $\phi = 1/6 \pi n \sigma^3$  ( $\sigma$ : particle diameter)

More realistic models use van-der Waals interactions, Coulomb interactions,...