Synchrotron Radiation Production and Properties
Part I: Radiation from Accelerated charged Particles

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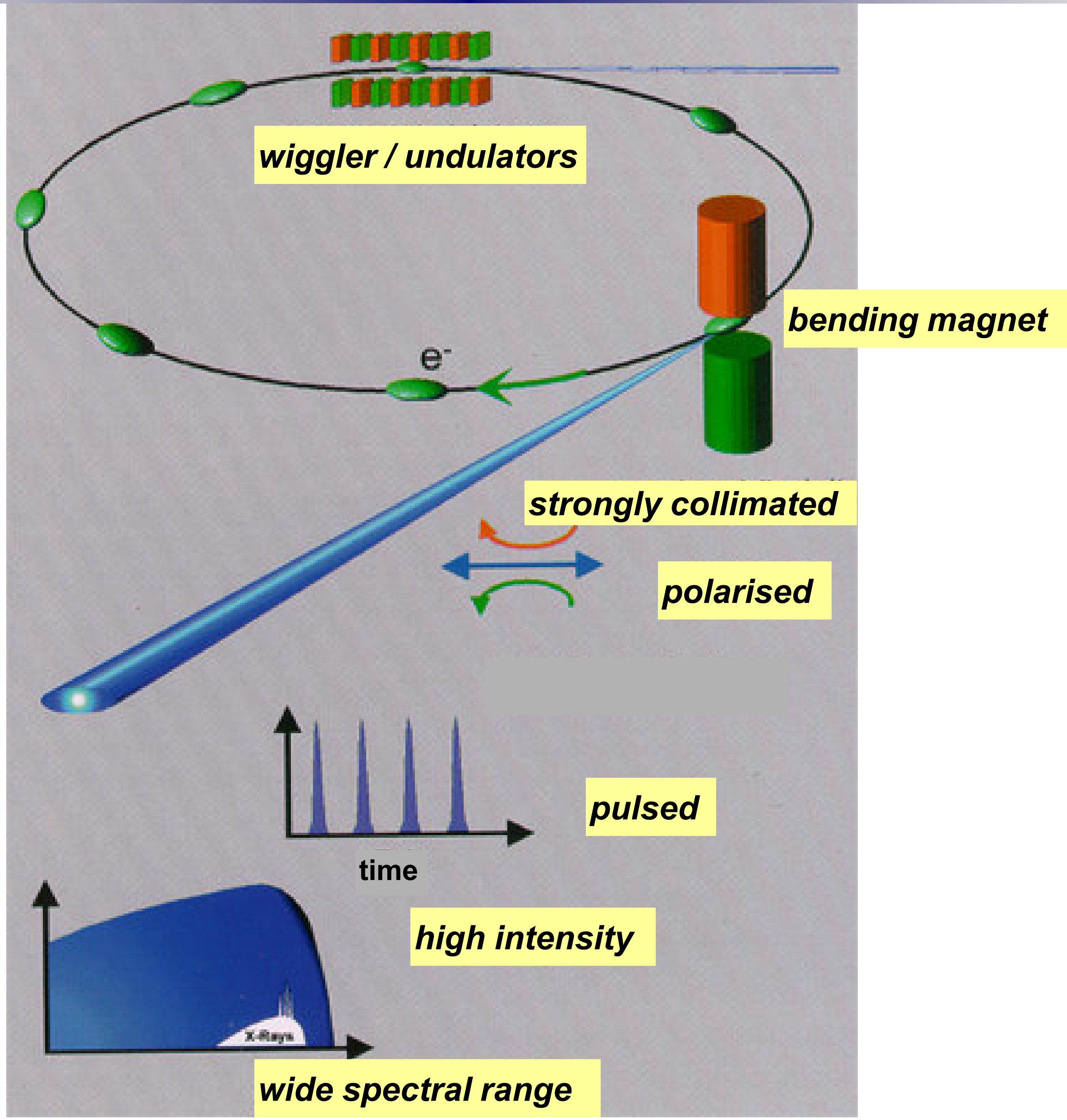
DESY Machine History

2000 Employees, 3000 International Guests
(100 apprentice, 100 undergraduate, 350 PHD, 300 Postdoc)
Annual Budget: 230 M€

DESY founded 1959 as an Electron Synchrotron Facility for Elementary Particle Research

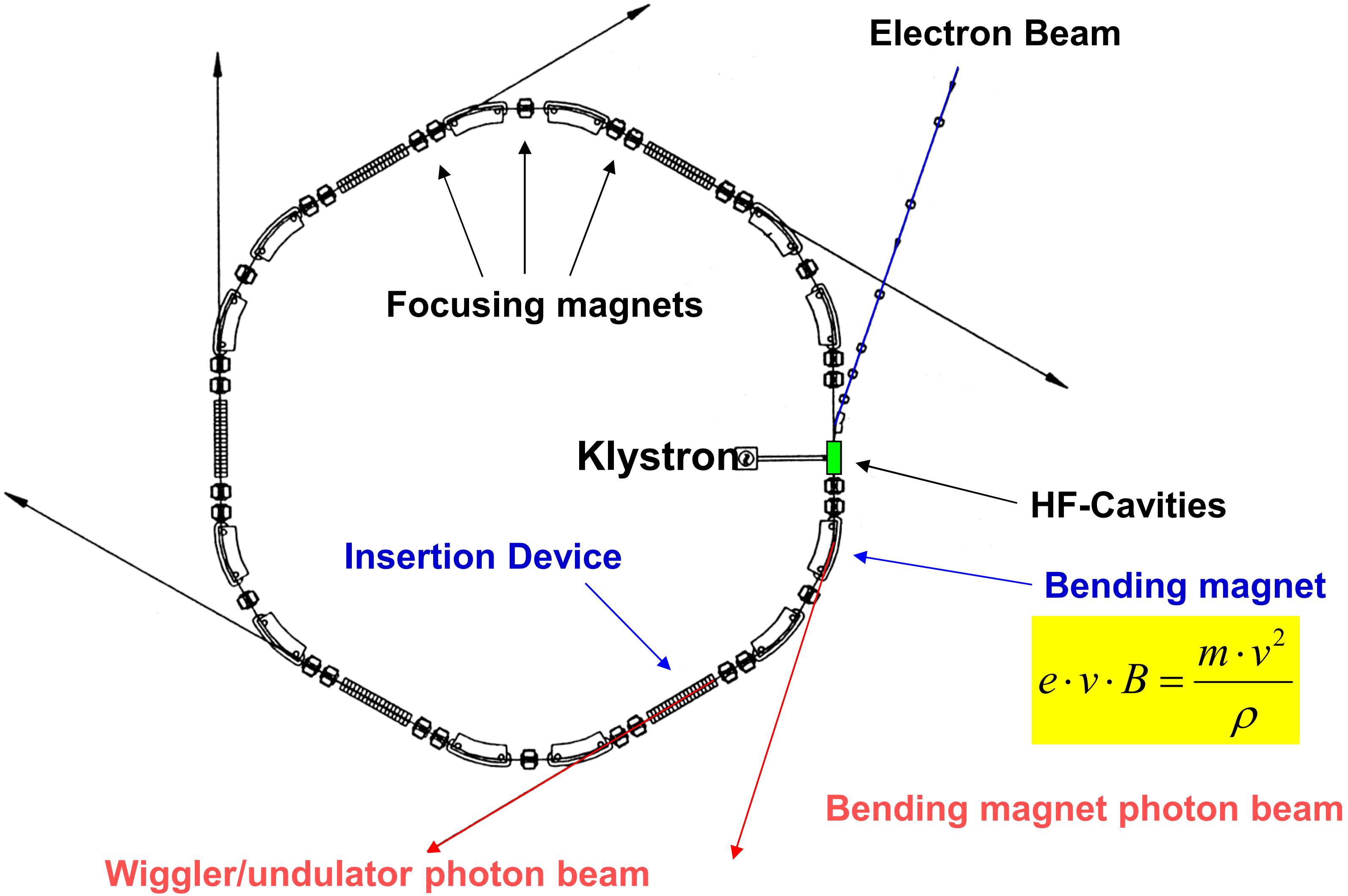
1964	DESY (Synchrotron)		e-	7.4 GeV
1974	DORIS (Storage Ring)	300m	e+/e-	3.5 GeV (later 5 GeV)
1980	HASYLAB@DORIS			
1984	Upgrade with 7 Wiggler/Undulator Beamlines			
1993	Dedicated SR Source at 4.5 GeV			
1978	PETRA (Storage Ring)	2.3km	e+/e-	19 GeV
1990	HERA (Storage Ring)	6.3km	p+/e-	920 GeV / 27.5 GeV (using PETRA as Booster)
1997	FLASH (Free Electron Laser)			
2005	Dedicated User Facility			
2007	Shutdown of HERA and Reconstruction of PETRA → PETRA III			
2009	PETRA III Dedicated SR Source at 6 GeV (presently most brilliant SR source worldwide)			
2012	Shutdown of DORIS			
2014	FLASH II (Extension of FLASH)			

Participation in the European XFEL project



wiggler / undulators
bending magnet
e⁻
strongly collimated
polarised
pulsed
time
high intensity
X-Rays
wide spectral range

Synchrotron Radiation from Storage Rings



$$e \cdot v \cdot B = \frac{m \cdot v^2}{\rho}$$

Maxwell Equations

Free space, SI-units

$$\vec{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Gauss's Law for E-Field

$$\vec{\nabla} \cdot \vec{E} \equiv div\,\vec{E} = \frac{\rho}{\varepsilon_0}$$

Charge is source of electrical field.
Electrical field lines diverge

Gauss's Law for B-Field

$$\vec{\nabla} \cdot \vec{B} \equiv div\,\vec{B} = 0$$

Magnetic field has no source
Magnetic field lines do not diverge

Faraday's Law of Induction

$$\vec{\nabla} \times \vec{E} \equiv rot\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Magnetic flux density changes
cause a closed electrical field (curl)

Ampere's circuital Law

$$\vec{\nabla} \times \vec{B} \equiv rot\,\vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \cdot \frac{\partial \vec{E}}{\partial t}$$

Electrical current and flux density changes
cause a closed magnetic field (curl)

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

ε_0 : free space permittivity
μ_0 : free space permeability

Electromagnetic Waves

Free Space, no charges, no currents:

$$\vec{\nabla}\cdot\vec{E}=0 \quad , \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \quad , \quad \vec{\nabla}\times\vec{B}=-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

Identity:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{X})=\vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{X})-\vec{\nabla}^2\vec{X}$$

Leads to:

$$\frac{1}{c^2}\cdot\frac{\partial^2\vec{E}}{\partial t^2}-\nabla^2\vec{E}=0 \quad and \quad \frac{1}{c^2}\cdot\frac{\partial^2\vec{B}}{\partial t^2}-\nabla^2\vec{B}=0$$

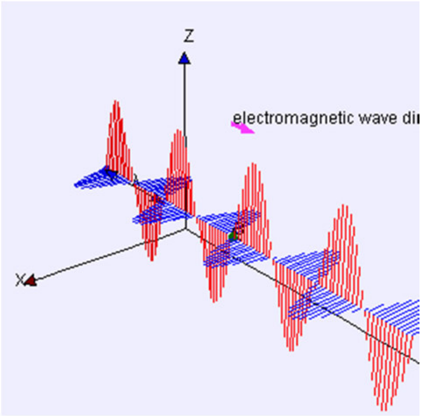
Solution: Plane Waves

$$\vec{E}=\vec{E}_0\sin(-\omega t+\vec{k}\cdot\vec{r})$$

$$\vec{B}=\vec{B}_0\sin(-\omega t+\vec{k}\cdot\vec{r})$$

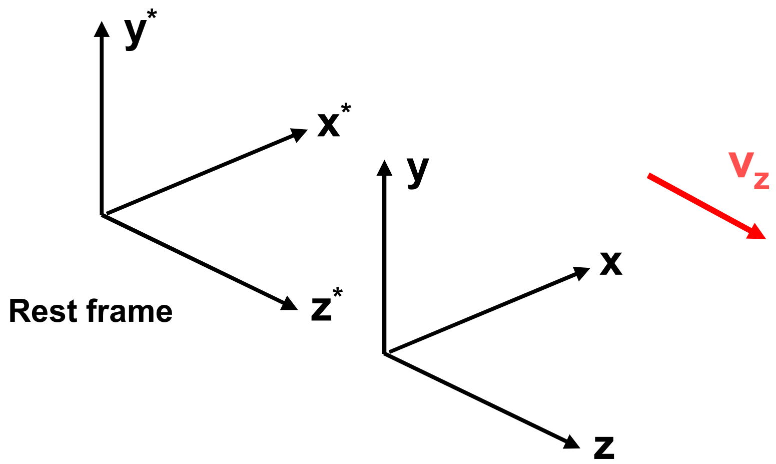
$$\vec{E}\perp\vec{B}\perp\hat{n}_{propagation}$$

$$\omega=\frac{2\pi}{\nu} \quad and \quad \vec{k}=\frac{2\pi}{\lambda}\cdot\hat{n}_{propagation}$$



Energy flow density (Pointing Vector) $\vec{S}=\frac{c}{4\pi}\cdot\left(\vec{E}\times\vec{B}\right)$

Gallilei-Transformation

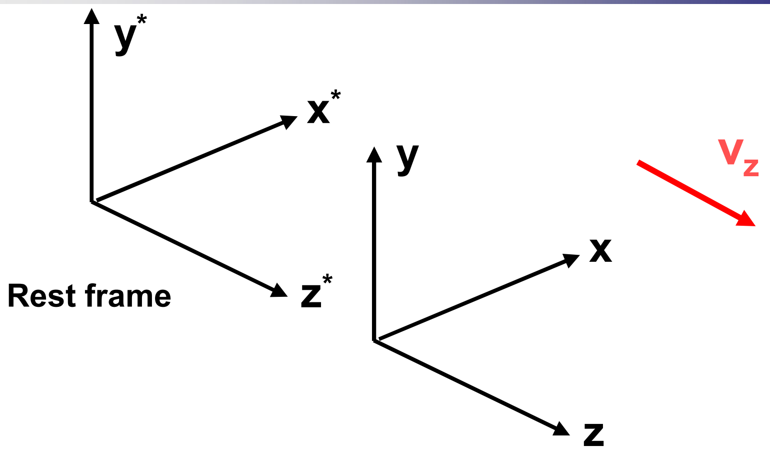


$$\begin{pmatrix} x^* \\ y^* \\ z^* \\ t^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -v_z \\ 0 & 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

Spatial vector: {x,y,z}
Time t

$$x^* = x$$
$$y^* = y$$
$$z^* = z - v_z t$$
$$t^* = t$$

Transformation of relativistic motions
Space-Time four-vector



$$x^* = x$$
$$y^* = y$$
$$z^* = \gamma(z - \beta_z ct)$$
$$ct^* = \gamma(ct - \beta_z z)$$

Lorentz-Transformation

$$\begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta_z\gamma \\ 0 & 0 & -\beta_z\gamma & \gamma \end{pmatrix} * \begin{pmatrix} x^* \\ y^* \\ z^* \\ ct^* \end{pmatrix}$$

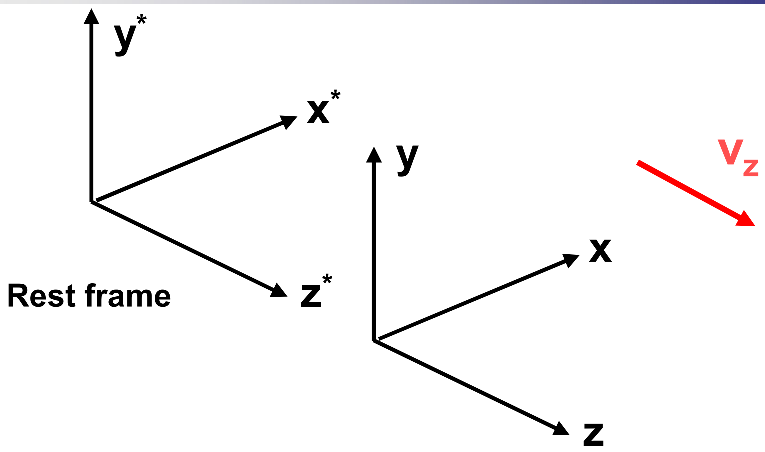
$$\beta_i = \frac{v_i}{c} \quad , \quad \gamma = \frac{1}{\sqrt{1-\beta_i^2}}$$

Time Dilation

$$dt = \gamma \cdot dt^*$$

Length Contraction

$$dx = \frac{dx^*}{\gamma}$$



Energy: $E = \gamma \cdot m_0 \cdot c^2$

Momentum: $\vec{p} = \gamma \cdot m_0 \cdot \vec{v}$

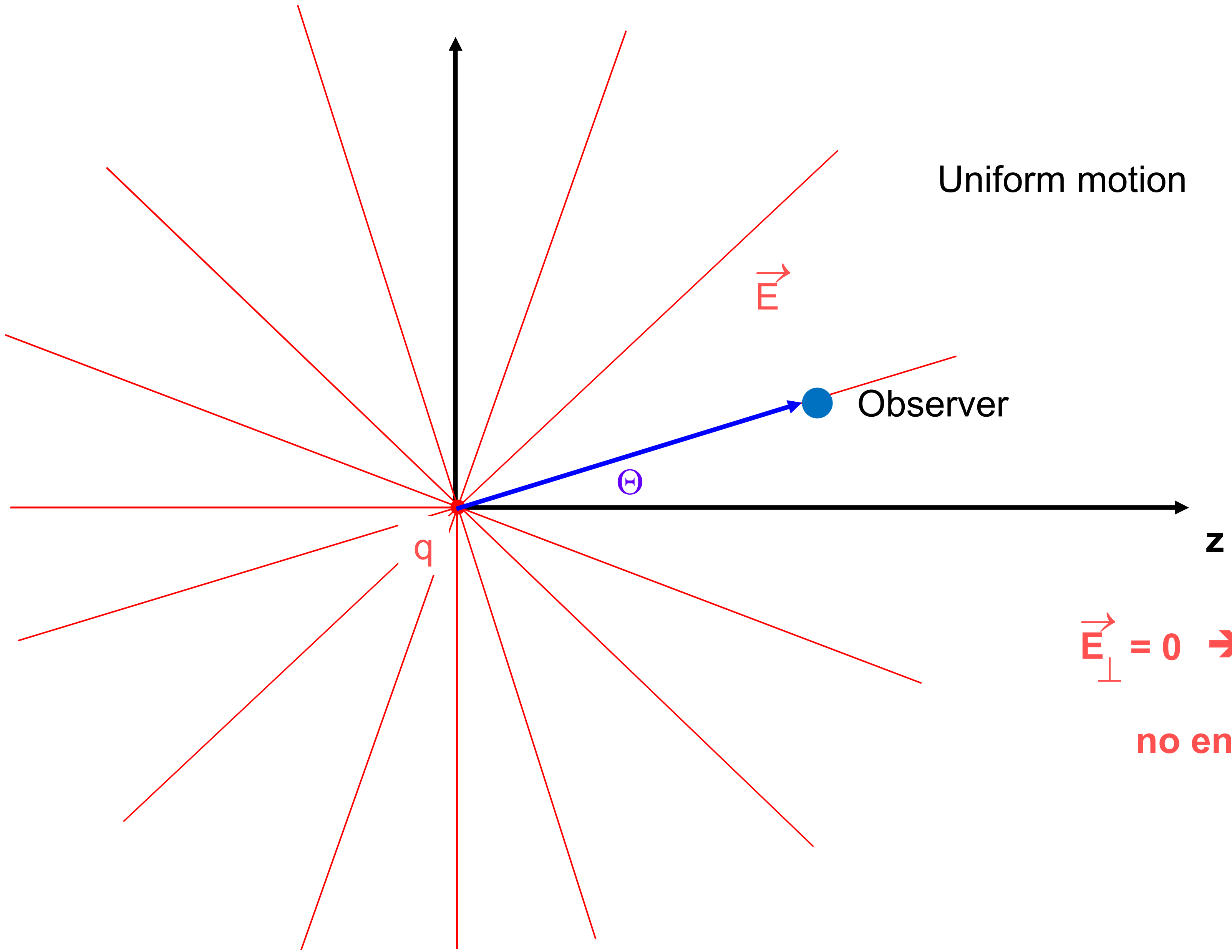
Lorentz-Transformation

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ \frac{E}{c} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & -\beta_z\gamma \\ 0 & 0 & -\beta_z\gamma & \gamma \end{pmatrix} * \begin{pmatrix} p_x^* \\ p_y^* \\ p_z^* \\ \frac{E^*}{c} \end{pmatrix}$$

➔ $$\vec{p}^{\,2} - \frac{E^2}{c^2} = -m_0^2 c^2$$

$$\beta_i = \frac{v_i}{c} \quad , \quad \gamma = \frac{1}{\sqrt{1-\beta_i^2}}$$

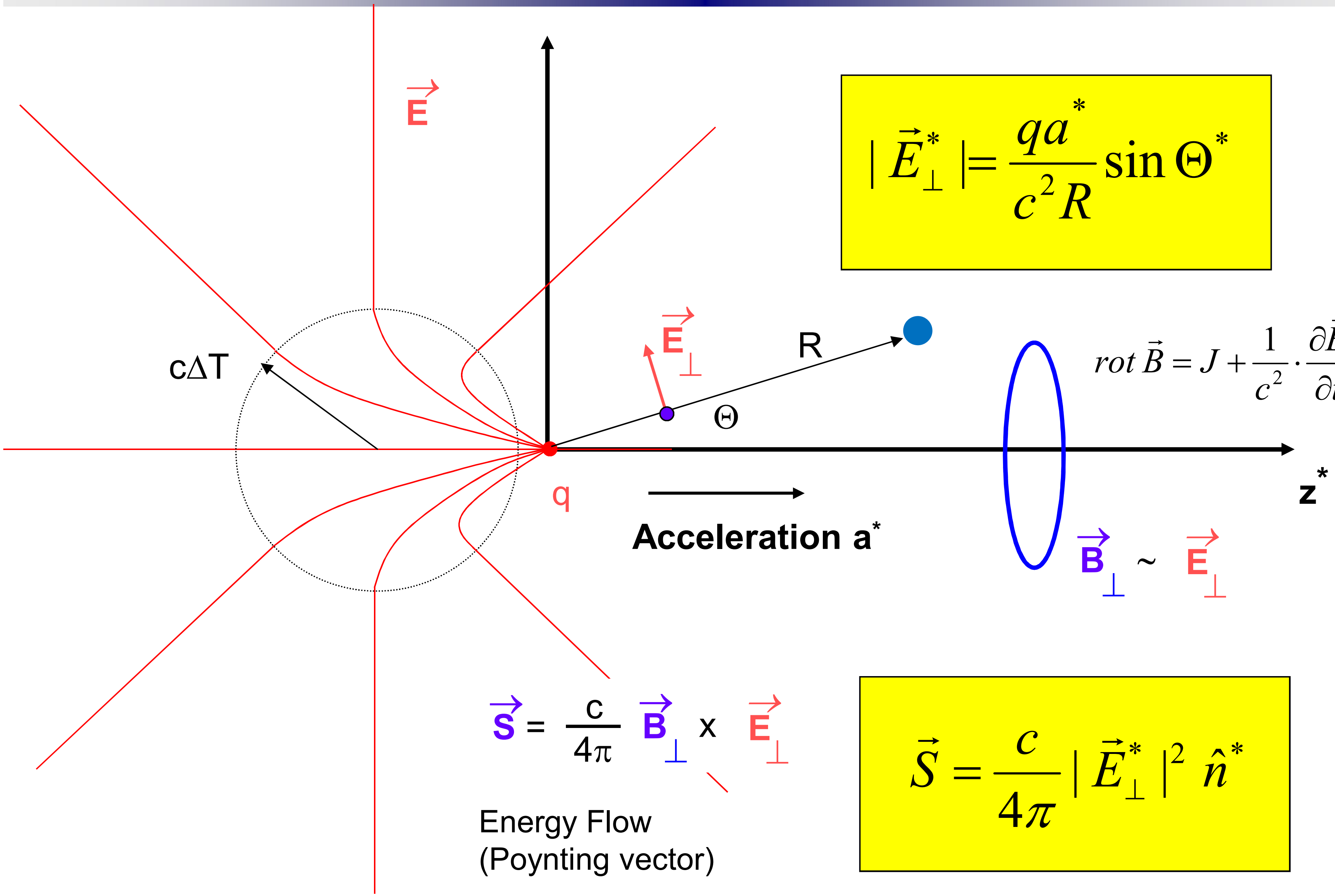
Static field of an electric charge



$\vec{E}_{\perp} = 0$ ➔

no energy flow

Longitudinal acceleration (parallel to z*)

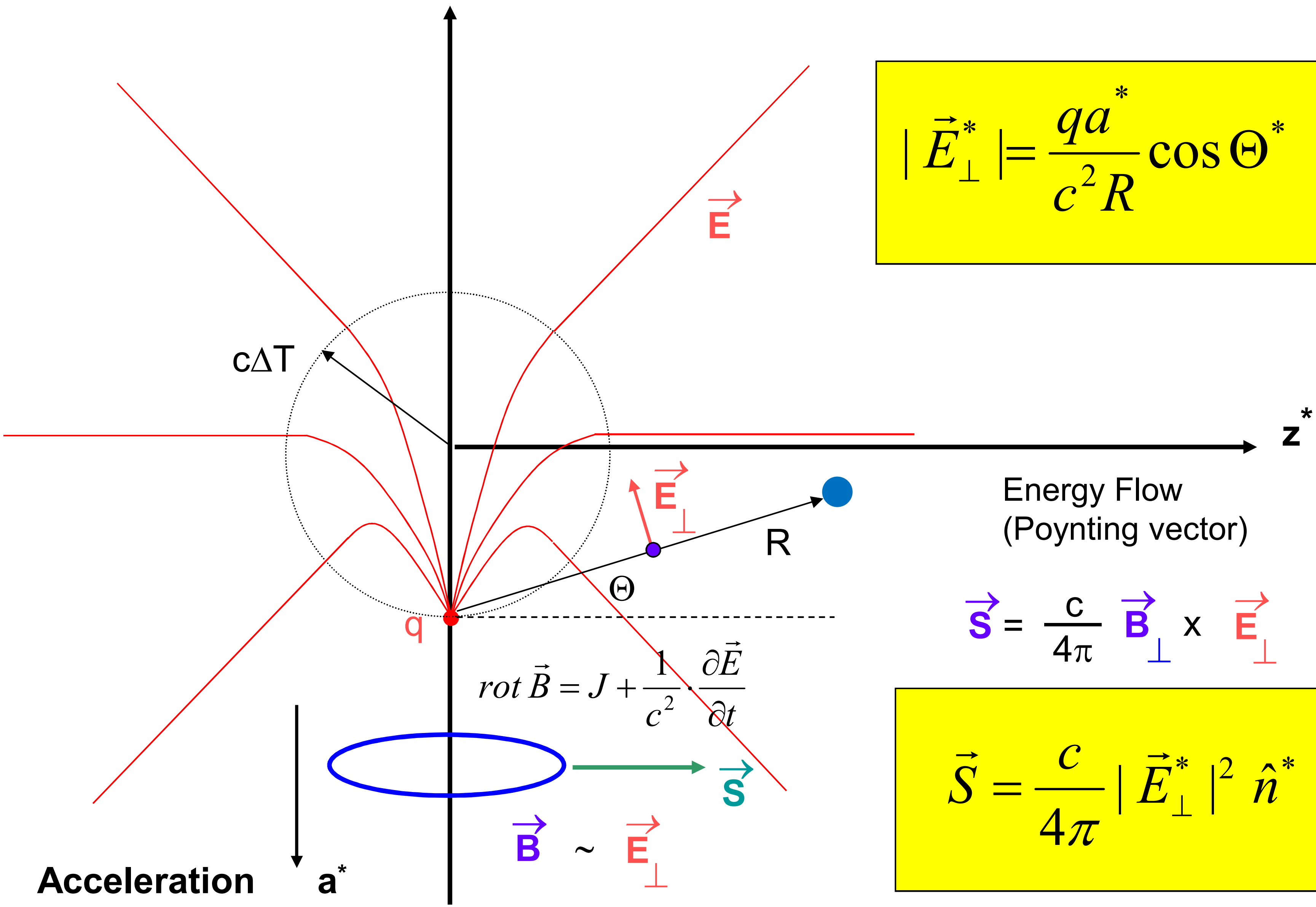


$$| \vec{E}^*_\perp | = \frac{q a^*}{c^2 R} \sin \Theta^*$$

$$rot\, \vec{B} = J + \frac{1}{c^2} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$\vec{S} = \frac{c}{4\pi} | \vec{E}^*_\perp |^2 \, \hat{n}^*$$

Transversal acceleration (perpendicular to z*)



$$|\vec{E}_\perp^*| = \frac{qa^*}{c^2 R} \cos\Theta^*$$

$$\vec{S} = \frac{c}{4\pi} |\vec{E}_\perp^*|^2 \, \hat{n}^*$$

Total Radiation Power

$$P = \int_\Omega \vec{S} dA^* = \int_\Omega \frac{c}{4\pi} \frac{q^2 a^{*2}}{c^4 R^2} \begin{Bmatrix} \sin^2 \Theta^* \\ \cos^2 \Theta^* \end{Bmatrix} \cdot \vec{n}^* \cdot d\vec{A}^*$$

$$a^* = \frac{\dot{p}^*}{m_0}$$

$$R^2 \sin \Theta^* d\Theta^* d\Psi^*$$

Spherical coordinates

$$P = \frac{2q^2}{3m_0^2 c^3} \cdot \left| \frac{d\vec{p}^*}{dt^*} \right|^2$$

Larmor Formula

Lorentz Invariant Representation of Lamor Formula

Lorentz invariant Momentum-Energy four-vector

$$dt = \gamma dt^*$$

$$\left\langle p_x, p_y, p_z, \frac{E}{c} \right\rangle \qquad \vec{p}^{\,2} - \frac{E^2}{c^2} = -m_0^2 c^2$$

$$P = \frac{2q^2}{3m_0^2 c^3} \cdot \left(\frac{d\vec{p}^{\,*}}{dt^*} \right)^2$$

$$P = \frac{2q^2}{3m_0^2 c^3} \cdot \gamma^2 \cdot \left[\left(\frac{d\vec{p}}{dt} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{dt} \right)^2 \right]$$

Linear (longitudinal acceleration along x)

$$E = \gamma m_0 c^2$$
$$p = \gamma m_0 v$$
$$\vec{p}^2 - \frac{E^2}{c^2} = -m_0^2 c^2$$

$$\Rightarrow \quad \frac{dE}{dt} = v \cdot \frac{dp}{dt}$$

because

$$E^2 = (m_0 c^2)^2 + p^2 c^2 \quad , t-derivation:$$
$$2E \cdot \frac{dE}{dt} = 2c^2 p \cdot \frac{dp}{dt} \quad , substitute\ E\ and\ p$$

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$$

$$\frac{dp}{dt} = \frac{1}{v} \cdot \frac{dE}{dx} \cdot \frac{dx}{dt} = \frac{dE}{dx}$$

$$P = \frac{2q^2}{3m_0^2 c^3} \cdot \gamma^2 \cdot \left[\left(\frac{d\vec{p}}{dt} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{dt} \right)^2 \right] \longrightarrow P_{linear} = \frac{2q^2}{3m_0^2 c^3} \left(\frac{dE}{dx} \right)^2$$

$$\frac{dp}{dt} = \frac{dE}{dx}$$

Example: Linear electron accelerator (Tesla) → dE/dx ~ 40 MeV/m

"Efficiency" $$\eta = \frac{P_{linear}}{dE/dt} = \frac{P_{linear}}{v\, dE/dx} = \frac{2e^2}{3m_e^2 c^4} \cdot \frac{dE}{dx} \sim 10^{-16}$$

$v \approx c$

Circular (transversal acceleration along z)

$$P = \frac{2q^2}{3m_0^2c^3} \cdot \gamma^2 \cdot \left[\left(\frac{d\vec{p}}{dt} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{dt} \right)^2 \right]$$

$\gamma = E / m_0c^2$

=0

$$\frac{dp}{dt} = p\omega = p\frac{v}{R} \approx p\frac{c}{R} \approx \frac{E}{R}$$

$$P_{circular} = \frac{2q^2}{3m_0^4c^7} \cdot \frac{E^4}{R^2}$$

Note: $$\frac{P_{electron}}{P_{proton}} = \left(\frac{m_p}{m_e} \right)^4 = 1.13 \cdot 10^{13}$$!

Decomposition of radiation pulse into plane wave components:

$$E^* = E_0^* \cdot e^{i(\omega^* t^* - \vec{k}^* \cdot \vec{r}^*)}$$

↑ phase

with

$$\vec{k}^* = \frac{2\pi}{\lambda^*}\hat{n}^* = \frac{\omega^*}{c}\hat{n}^*$$

$$\omega^*[ct^* - n_x^* x^* - n_y^* y^* - n_z^* z^*] = \omega[ct - n_x x - n_y y - n_z z]$$

$$= \omega^*[\gamma ct - \gamma\beta_z z - n_x^* x - n_y^* y - n_z^* \gamma z + n_z^* \gamma\beta_z ct]$$

$$x^* = x$$
$$y^* = y$$
$$z^* = \gamma(z - \beta_z ct)$$
$$ct^* = \gamma(ct - \beta_z z)$$

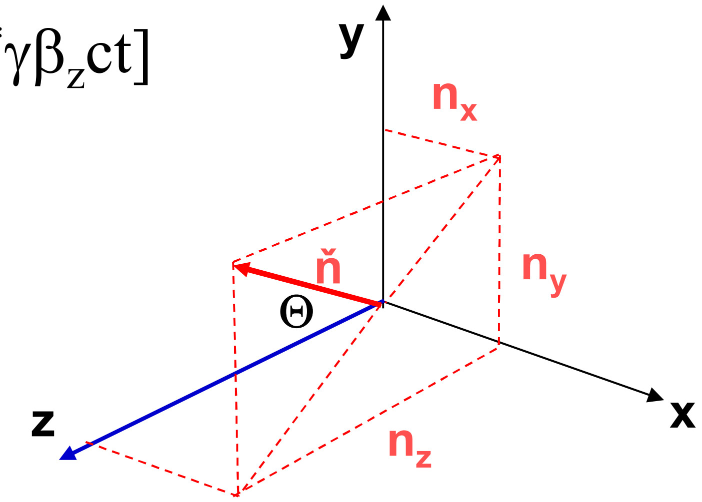
$$\omega = \omega^* \gamma[1 + n_z^* \beta_z]$$

Relativistic Doppler Effect

Spatial Characteristics of Synchrotron Radiation II

$$\omega^*[ct^* - n_x^* x^* - n_y^* y^* - n_z^* z^*] = \omega[ct - n_x x - n_y y - \mathbf{n_z z}]$$

$$= \omega^*[\gamma ct - \gamma\beta_z \mathbf{z} - n_x^* x - n_y^* y - \mathbf{n_z^* \gamma z} + n_z^* \gamma\beta_z ct]$$



$$n_x = \frac{n_x^*}{\gamma(1+\beta_z n_z^*)}$$

$$n_y = \frac{n_y^*}{\gamma(1+\beta_z n_z^*)}$$

$$n_z = \frac{\beta_z + n_z^*}{(1+\beta_z n_z^*)}$$

$$n_x^2 + n_y^2 = \sin^2\Theta$$

$$n_z = \cos\Theta$$

$$n_x^{*2} + n_y^{*2} = \sin^2\Theta^*$$

$$n_z^* = \cos\Theta^*$$

$$\rightarrow \quad \sin\Theta = \frac{\sin\Theta^*}{\gamma(1+\beta_z\cos\Theta^*)}$$

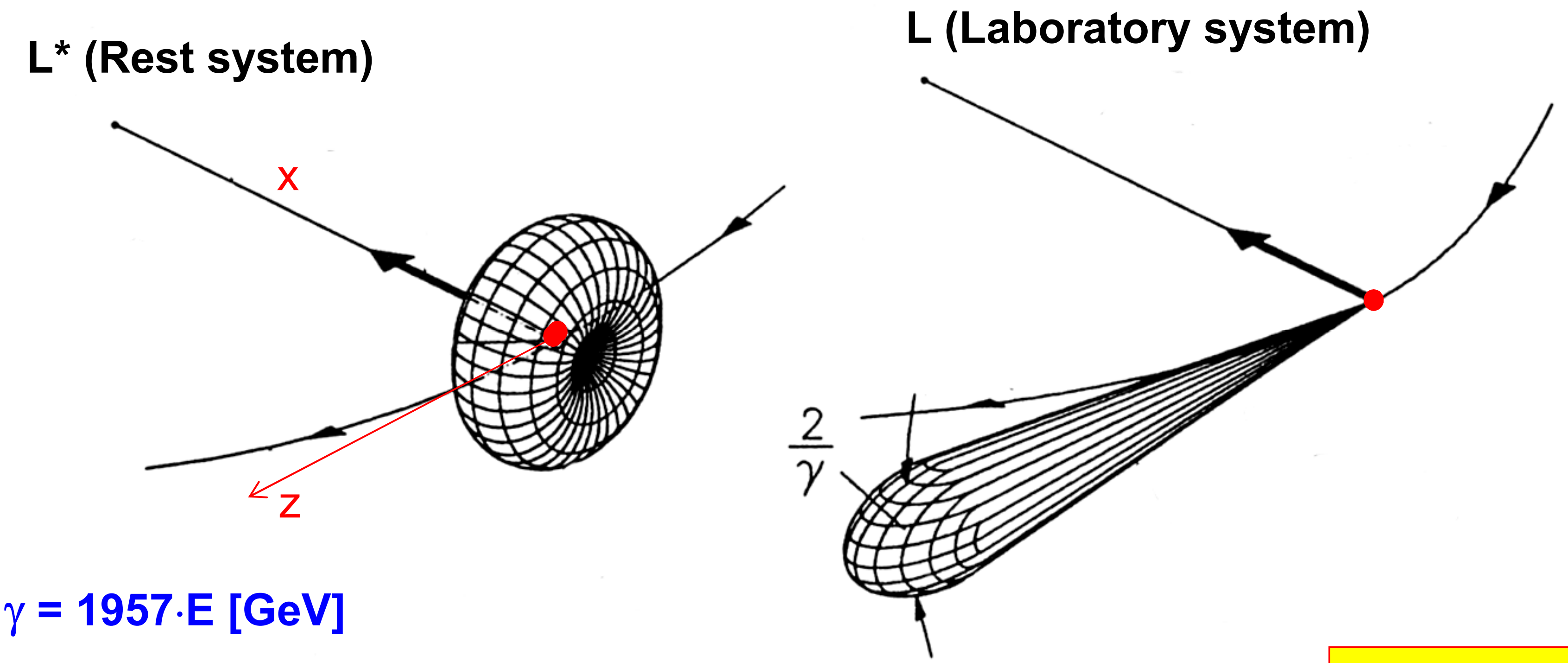
$\Theta^* = 90^o$

$$\Theta \approx \frac{1}{\gamma}$$

Z: direction of motion
X: direction of acceleration
Θ: angle between given direction and direction of motion (Z)

Angular distribution of emitted radiation

$$-\frac{\pi}{2} \leftrightarrow +\frac{\pi}{2} \quad \longrightarrow \quad -\frac{1}{\gamma} \leftrightarrow +\frac{1}{\gamma}$$



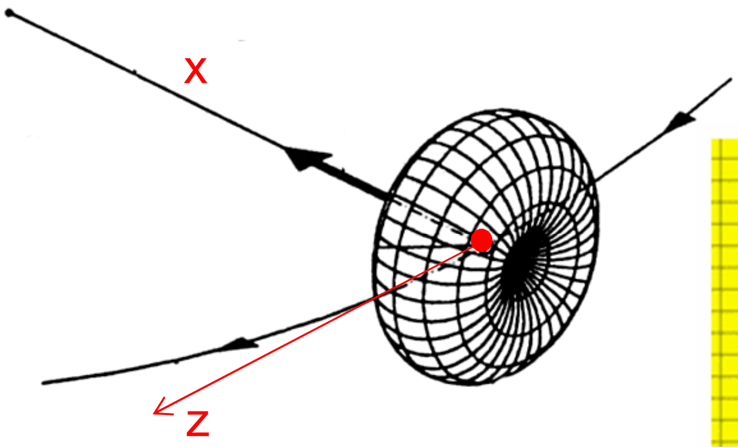
γ = 1957·E [GeV]

Example PETRA III:
E = 6.0 GeV → γ = 11742 → Θ = 0.0085 mrad = 0.005°

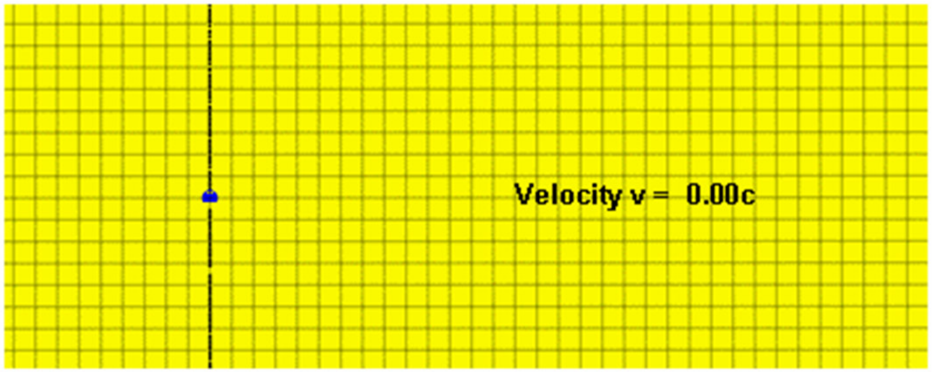
$$\frac{2}{\gamma} = \frac{2m_0c^2}{E}$$

$$-\frac{\pi}{2} \leftrightarrow +\frac{\pi}{2} \quad \longrightarrow \quad -\frac{1}{\gamma} \leftrightarrow +\frac{1}{\gamma}$$

L* (Rest system)



L (Laboratory system)



γ = 1957·E [GeV]

Example PETRA III:
E = 6.0 GeV → γ = 11742 → Θ = 0.0085 mrad = 0.005°

$$\frac{2}{\gamma} = \frac{2m_0c^2}{E}$$

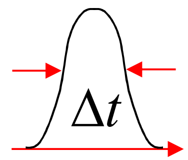
Time structure of radiation emitted by a single particle

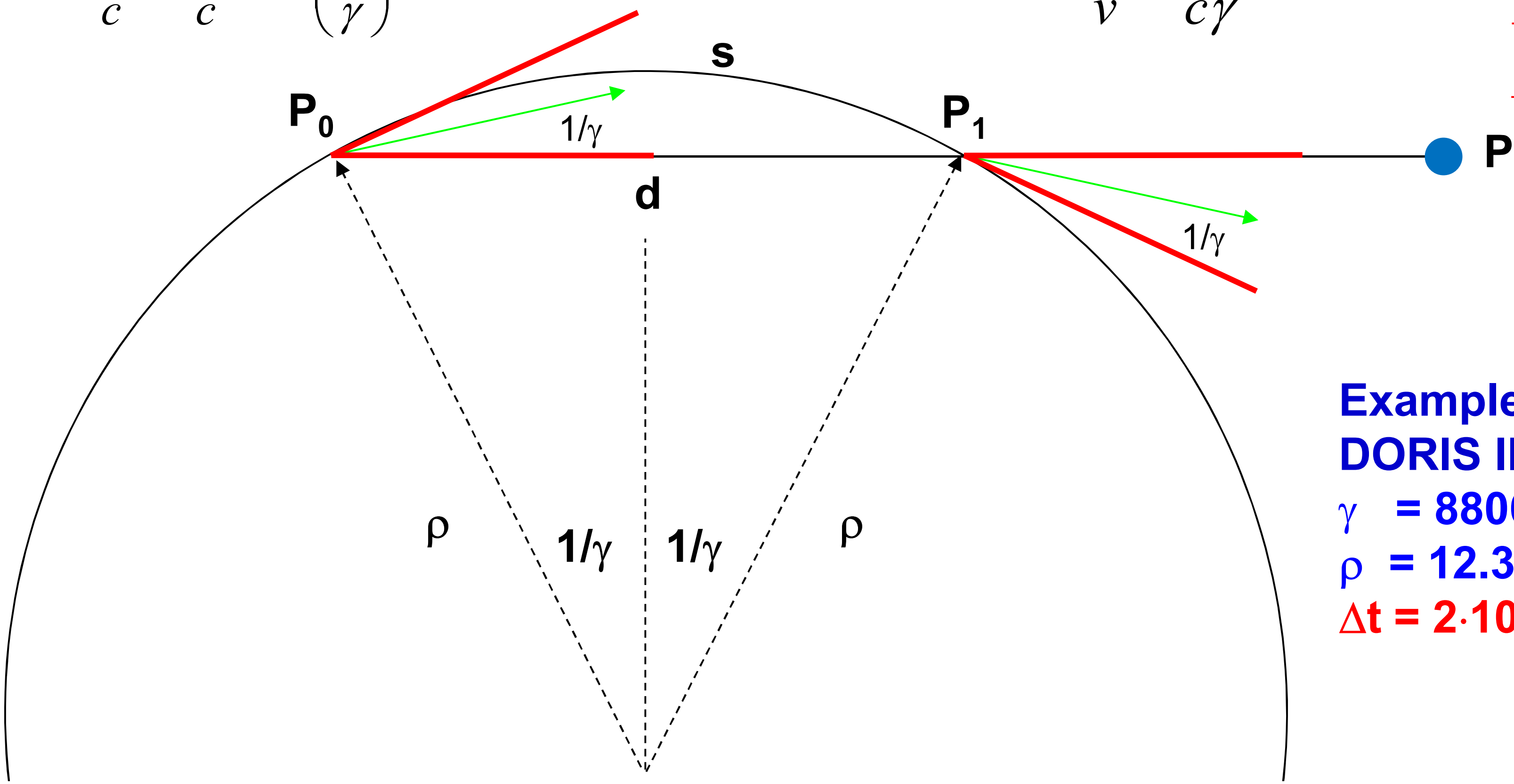
$$\Delta t = t_2 - t_1 = \frac{2\rho}{c}\left[\frac{1}{\gamma} - \sin\left(\frac{1}{\gamma}\right)\right] = \frac{\rho}{3c\gamma^3}$$

Taylor: $\approx \frac{1}{\gamma} - \frac{1}{6\gamma^3}$

$$t_1 = \frac{d}{c} = \frac{2\rho}{c}\cdot\sin\left(\frac{1}{\gamma}\right)$$

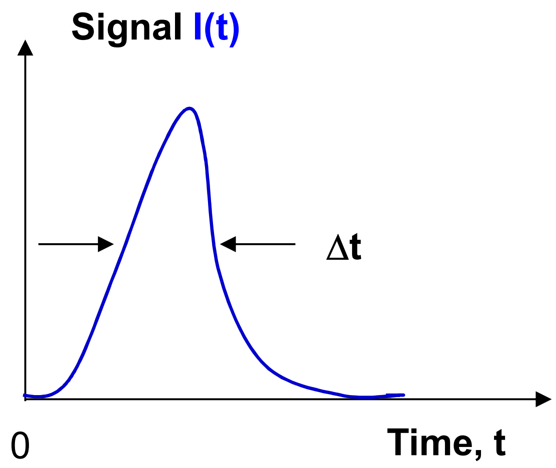
$$t_2 = \frac{s}{v} \approx \frac{2\rho}{c\gamma}$$, v≈c





Example:
DORIS III
γ = 8806
ρ = 12.3 m
$\Delta t = 2 \cdot 10^{-20}$ s

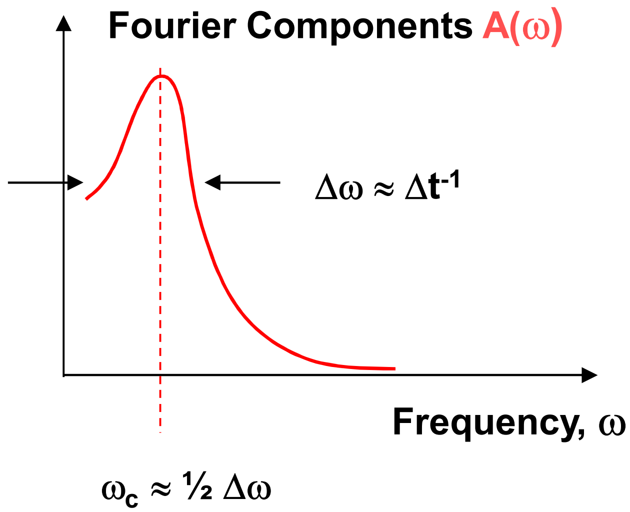
Fourier Composition



$$I(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(\omega) \cdot e^{i\omega t} d\omega$$

$$\updownarrow$$

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} I(t) \cdot e^{-i\omega t} dt$$



$$\omega_c \approx \frac{1}{\frac{1}{2}\Delta t} \approx \frac{6c\gamma^3}{\rho} \quad \Rightarrow \quad \varepsilon_c = \hbar \frac{6c}{(m_0 c^2)^3} \frac{E^3}{\rho}$$

A convenient formula for the critical energy

Critical Energy $$\varepsilon_c = \hbar \frac{6c}{(m_0 c^2)^3} \frac{E^3}{\rho}$$

with $$\rho[m] = \frac{3.335}{B[T]} \cdot E[GeV]$$

Central Force = Lorentz Force

$$e \cdot v \cdot B = \frac{m \cdot v^2}{\rho} = \frac{E \cdot v}{c \cdot \rho}$$

$$\Rightarrow \quad \rho = \frac{E}{e \cdot c \cdot B}$$

gives $$\boxed{\varepsilon_c = 0.665 \cdot E^2[GeV] \cdot B[T]}$$

Example PETRA III

Particle Energy	E = 6.0 GeV
Curvature radius of bending magnets	ρ = 22.92 m
Magnetic field of bending magnets	B = 0.873 T
Critical photon energy from bending magnets	20.9 keV ≡ 0.06 nm

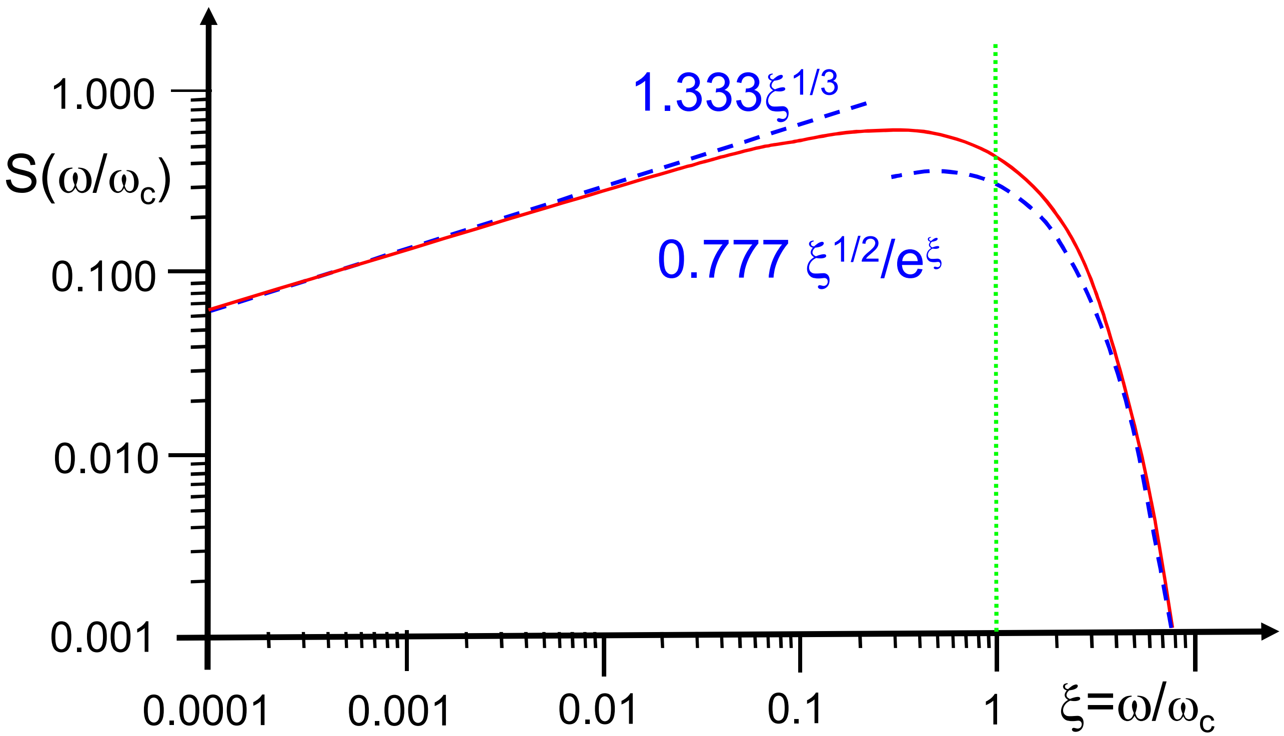
Spectral photon density of a bending magnet

$$\frac{d\dot{N}}{d\varepsilon/\varepsilon} = I_{beam}\frac{P_0}{\hbar\omega_c}S\left(\frac{\omega}{\omega_c}\right) \quad ; \qquad S(\xi) = \frac{9\sqrt{3}}{8\pi}\xi\cdot\int_{\xi}^{\infty}K_{5/3}(\xi)d\xi$$

Bessel Function

$$\xi := \frac{\omega}{\omega_c}$$

$$\int_0^1 S(\xi)d\xi = \frac{1}{2}$$



Summary: Transversely accelerated relativistic charge

1. Emitted radiation power $\propto \dfrac{E^4}{m_0^4 \cdot R^2}$

2. Observed spectrum shifted to higher frequencies due to Lorentz Transformation (Relativistic Doppler Effect)

3. Wide frequency spectrum due to short pulse duration

4. Radiation mainly observed in the forward direction due to Lorentz Transformation

PETRA III Parameters

Positron energy (E)	6		GeV
Maximum positron beam current (I)	100		mA
Circumference	2304		m
Number of bunches	40	/ 960	
Bunch Length	100		psec
Revolution time	7.6		µsec
Bunch Separation	190	/ 8	nsec
Maximum beam lifetime	2	/ 24	h
Horizontal positron beam emittance (ε_x)	1		nmrad
Coupling factor	1		%
Vertical positron beam emittance (ε_y)	0.01		nmrad
Positron beam energy spread (rms)	0.11		%
Curvature radius of bending magnets	22.92		m
Magnetic field of bending magnets	0.873		T
Critical photon energy from bending magnets	20.9		keV
Radiation Power at Dipole	52.3		kW

Ensemble of Electrons in a Storage Ring

Electron Energy E,
Dipole Field B, Bending radius R → **Photon Intensity I(Θ,Ψ,ω)**
Beam Current I_b

HF Cavities → Electron Bunches → **Pulsed time structure**

Quadrupoles, Sextupoles (Focusing) → **Emittance ε_i and β-function →**
Electron beam size σ_i and divergence σ'_i

$$\varepsilon_x = \sigma_x \sigma'_x \quad , \quad \varepsilon_y = \sigma_y \sigma'_y$$

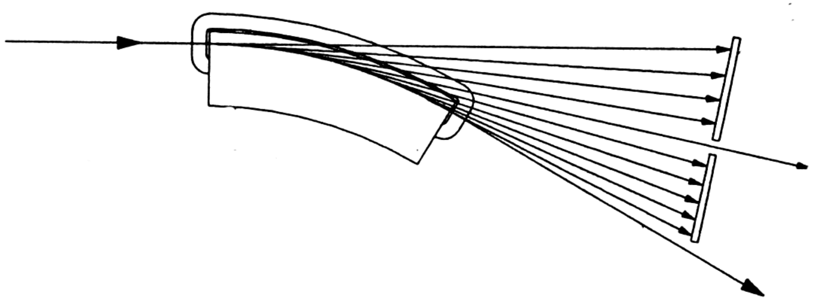
$$\sigma_x = \sqrt{\varepsilon_x \beta_x} \quad , \quad \sigma_y = \sqrt{\varepsilon_y \beta_y}$$

$$\sigma'_x = \varepsilon_x / \sigma_x \quad , \quad \sigma'_y = \varepsilon_y / \sigma_y$$

$$\sigma_{i,photon} = \sqrt{\sigma^2_{i,electron} + \sigma^2_{i,ph}}$$

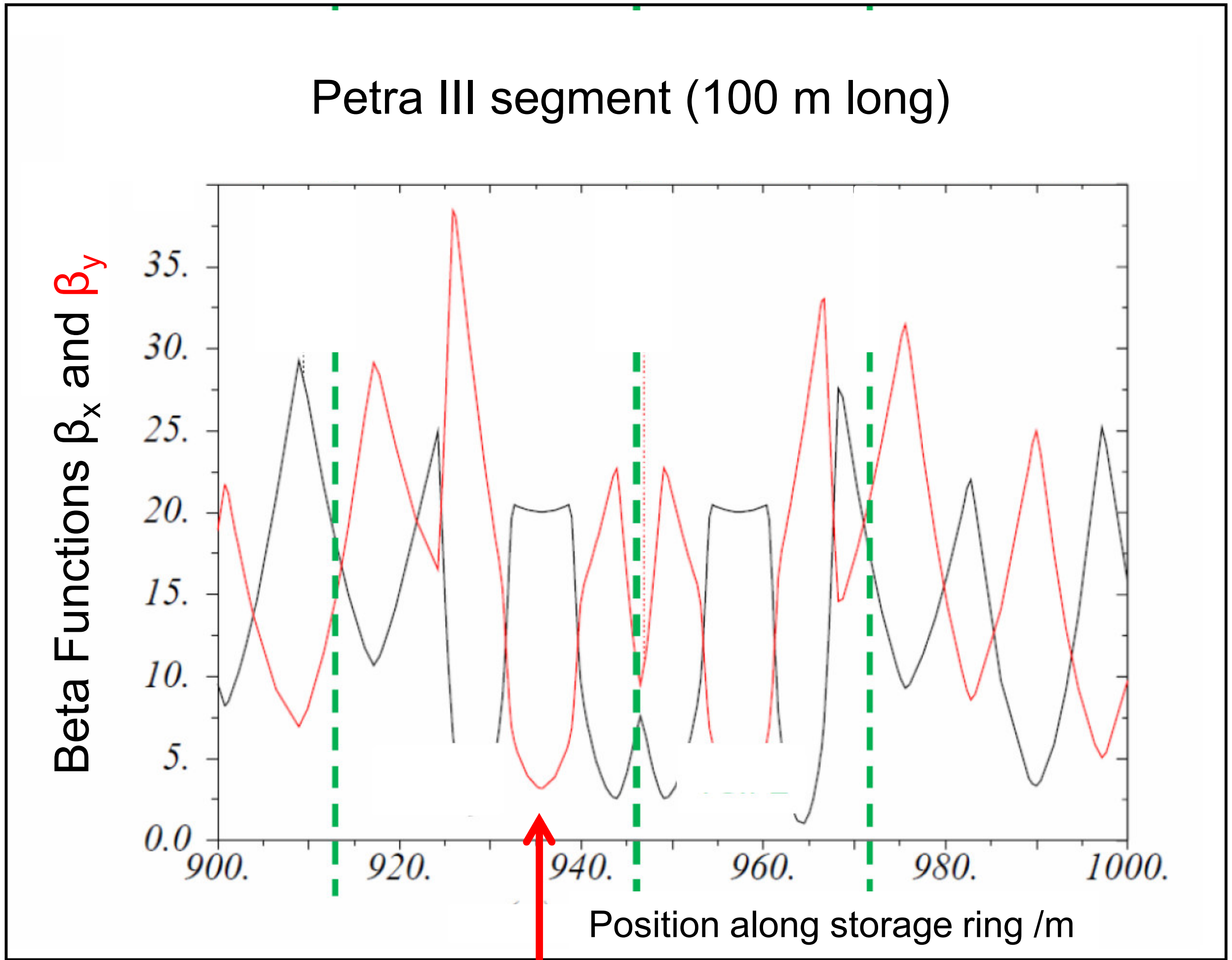
rms-values

Dipol (Bending magnet)



Horizontal divergence defined by aperture slits

Emittance, Beta Function, Beam Size & Divergence



PETRA III Emittance:

ε_x = 1 nm rad

ε_y = 0.01 nm rad

β_x = 20 m

β_y = 3 m

Horizontal beam size σ_x	0.141	mm
Horizontal beam divergence σ'_x	0.0071	mrad
Vertical beam size σ_y	0.00548	mm
Vertical beam divergence σ'_y	0.00182	mrad

Different quantities to describe photon intensity

Total Flux *F*
number of photons
per time

$$[F_{tot}] = \frac{Number\ of\ photons}{s}$$

Spectral Flux
number of photons
per time and energy

$$[F] = \frac{Number\ of\ photons}{s \cdot 0.1\%BW}$$

Brilliance *B*
number of photons
per time, energy, solid angle
and source area

$$[B] = \frac{Number\ of\ photons}{s \cdot mm^2 \cdot mrad^2 \cdot 0.1\%BW}$$

Peak brilliance B^{peak}
brilliance scaled to total pulse duration

$$B^{peak} = \frac{B}{\tau \times f}$$

τ - pulse duration
f - pulse frequency